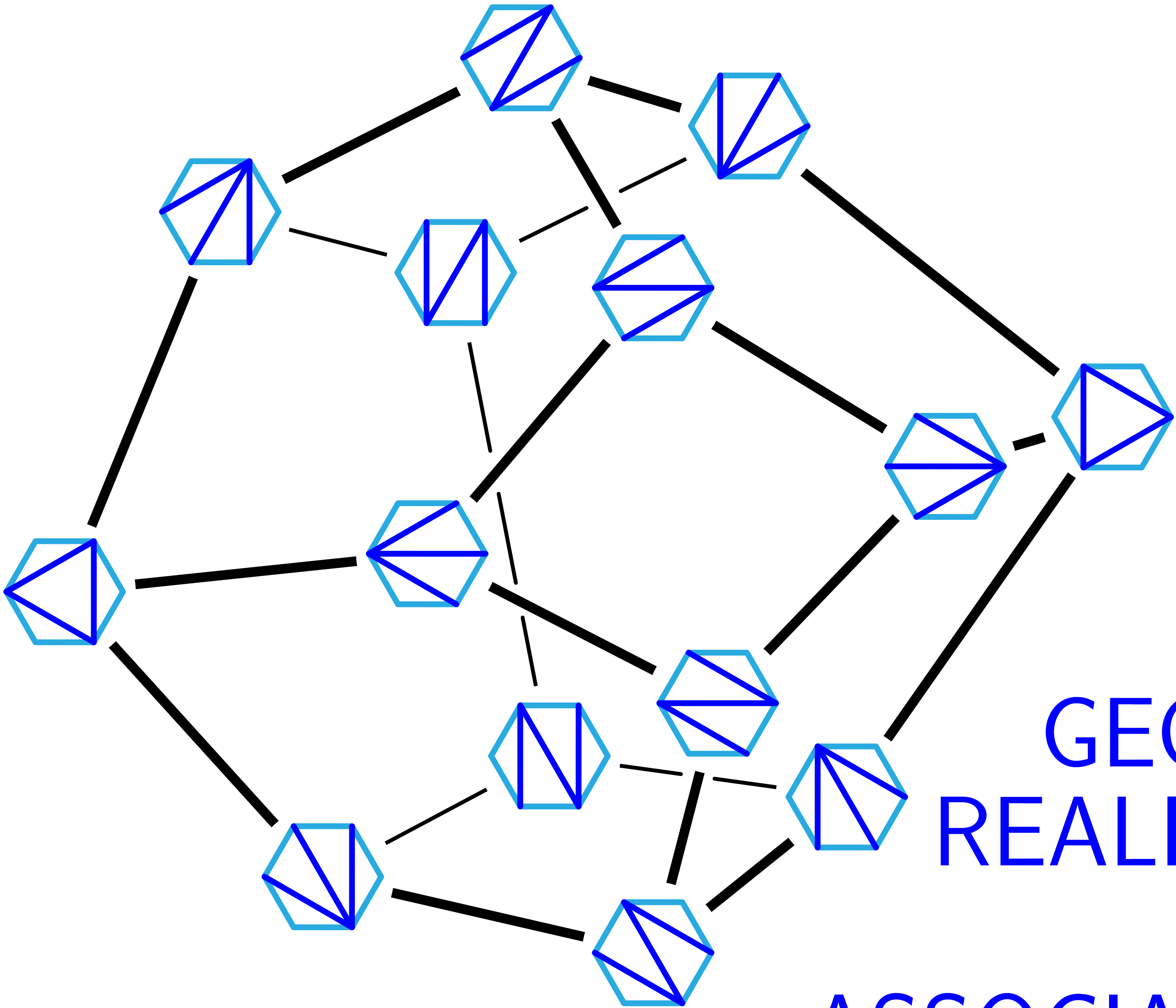


CombinatoireS
July 2, 2015

V. PILAUD
(CNRS & LIX)

MANY GEOMETRIC REALIZATIONS OF THE ASSOCIAHEDRON

POLYTOPES & COMBINATORICS

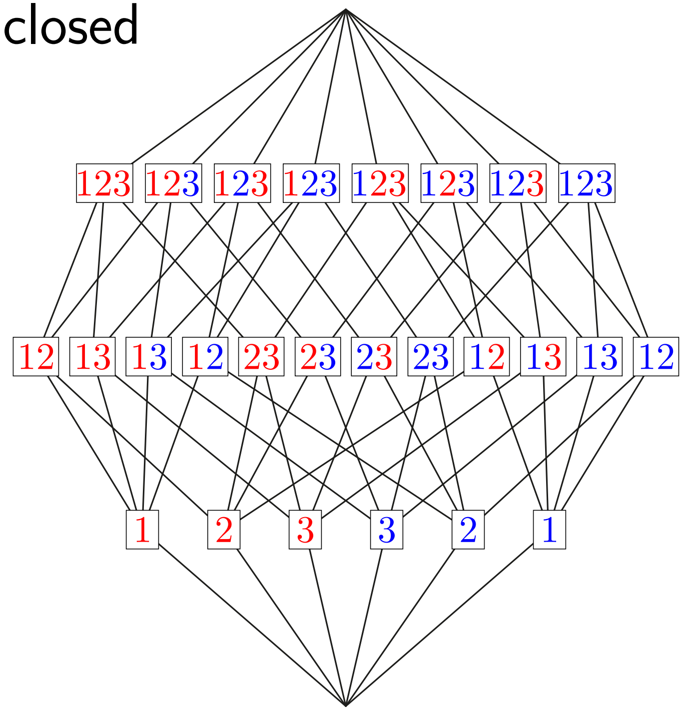
SIMPLICIAL COMPLEX

simplicial complex $=$ collection of subsets of X downward closed

exm:

$$X = [n] \cup [n]$$

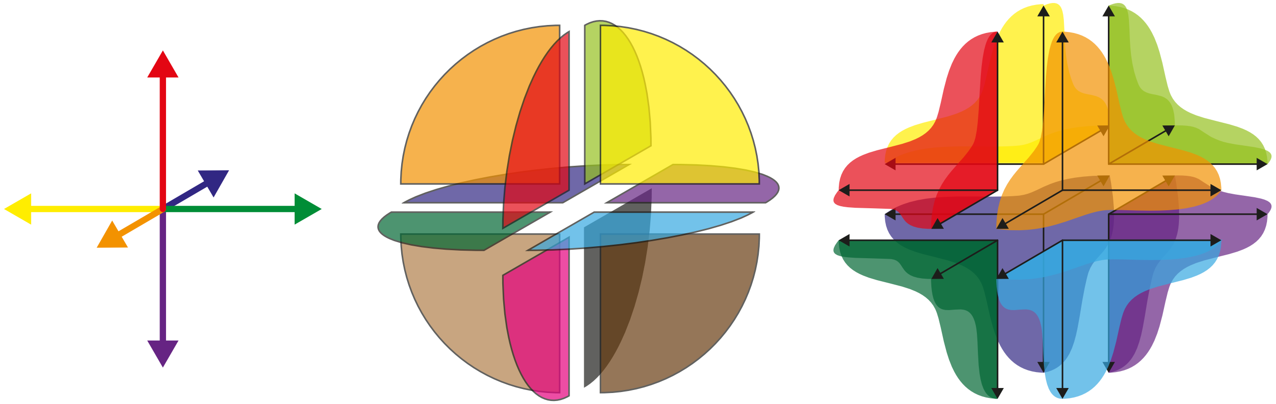
$$\Delta = \{I \subseteq X \mid \forall i \in [n],\ \{i, i\} \not\subseteq I\}$$

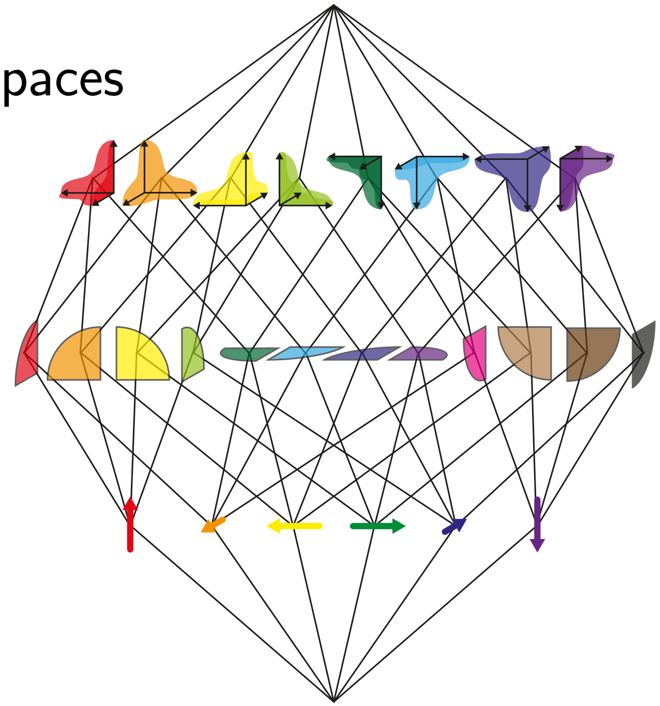


FANS

polyhedral cone $=$ positive span of a finite set of $\mathbb{R}^d$
$=$ intersection of finitely many linear half-spaces

fan $=$ collection of polyhedral cones closed by faces
and where any two cones intersect along a face





simplicial fan $=$ maximal cones generated by d rays

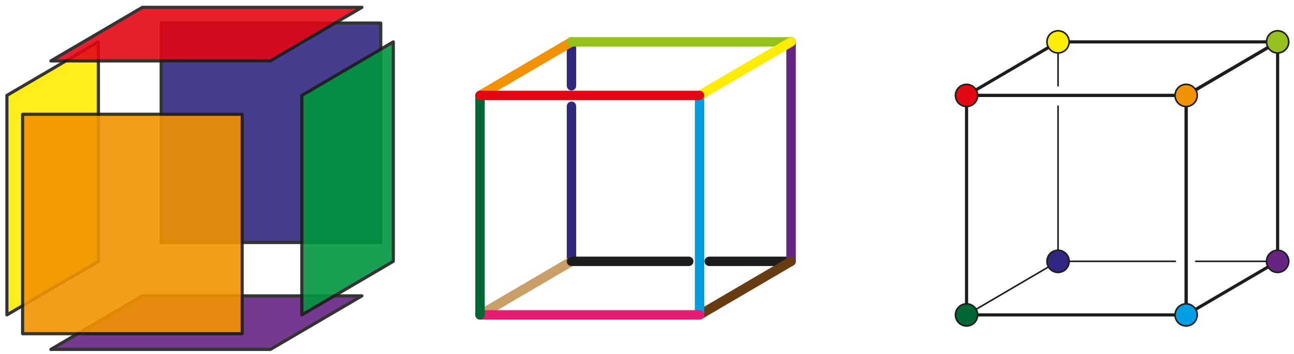
POLYTOPES

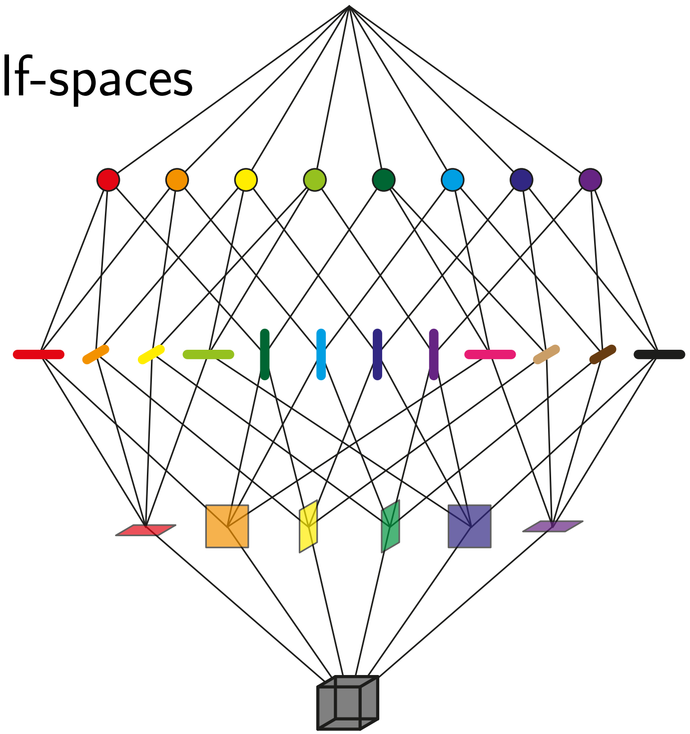
polytope = convex hull of a finite set of $\mathbb{R}^d$

= bounded intersection of finitely many affine half-spaces

face = intersection with a supporting hyperplane

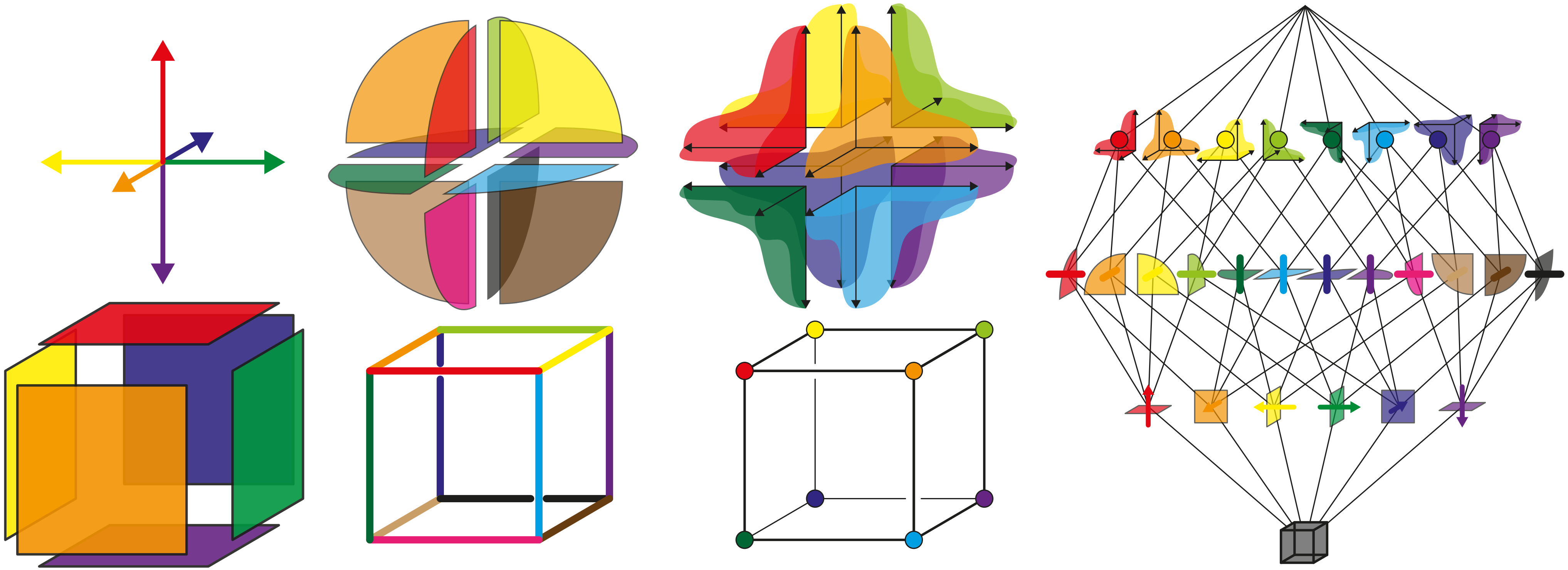
face lattice = all the faces with their inclusion relations





simple polytope = facets in general position = each vertex incident to d facets

SIMPLICIAL COMPLEXES, FANS, AND POLYTOPES



P polytope, F face of P

normal cone of F = positive span of the outer normal vectors of the facets containing F

normal fan of P = $\{$ normal cone of F $|$ F face of P $\}$

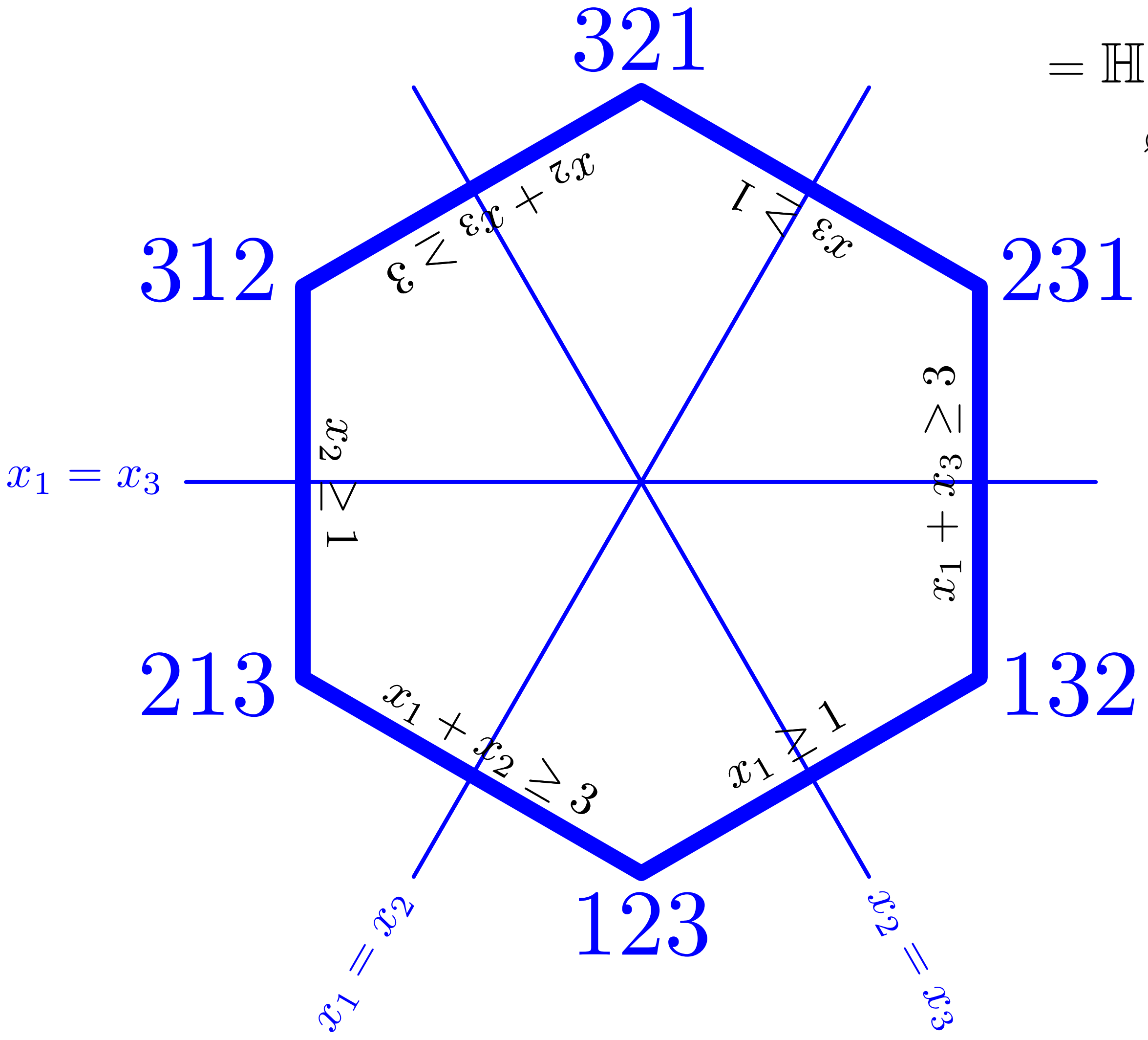
simple polytope $\Longrightarrow$ simplicial fan $\Longrightarrow$ simplicial complex

PERMUTAHEDRON

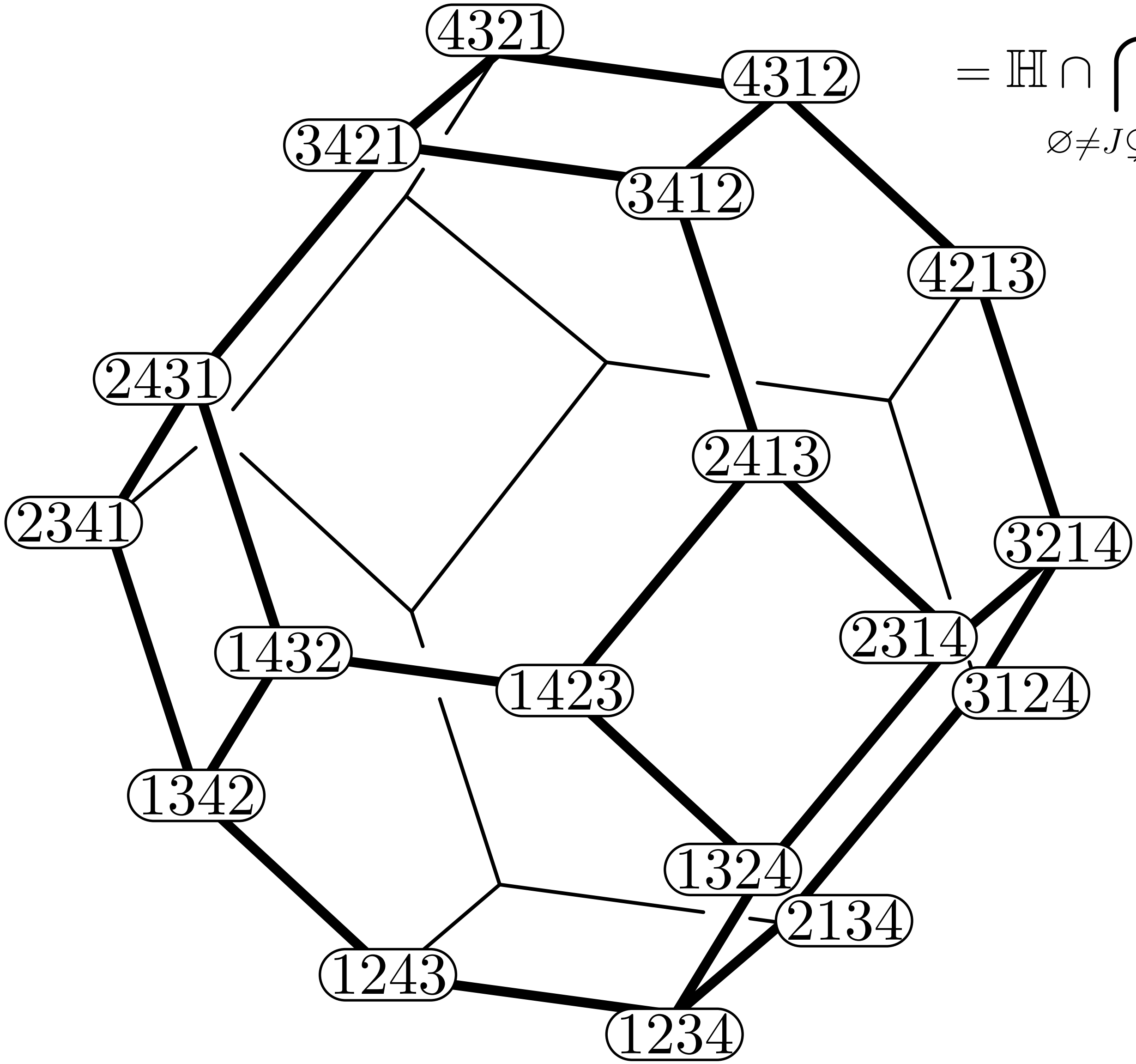
Permutohedron $\mathsf{Perm}(n)$

$$= \operatorname{conv}\{(\sigma(1), \dots, \sigma(n+1)) \mid \sigma \in \Sigma_{n+1}\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n+1]} \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j \in J} x_j \geq \binom{|J|+1}{2} \right\}$$



PERMUTAHEDRON



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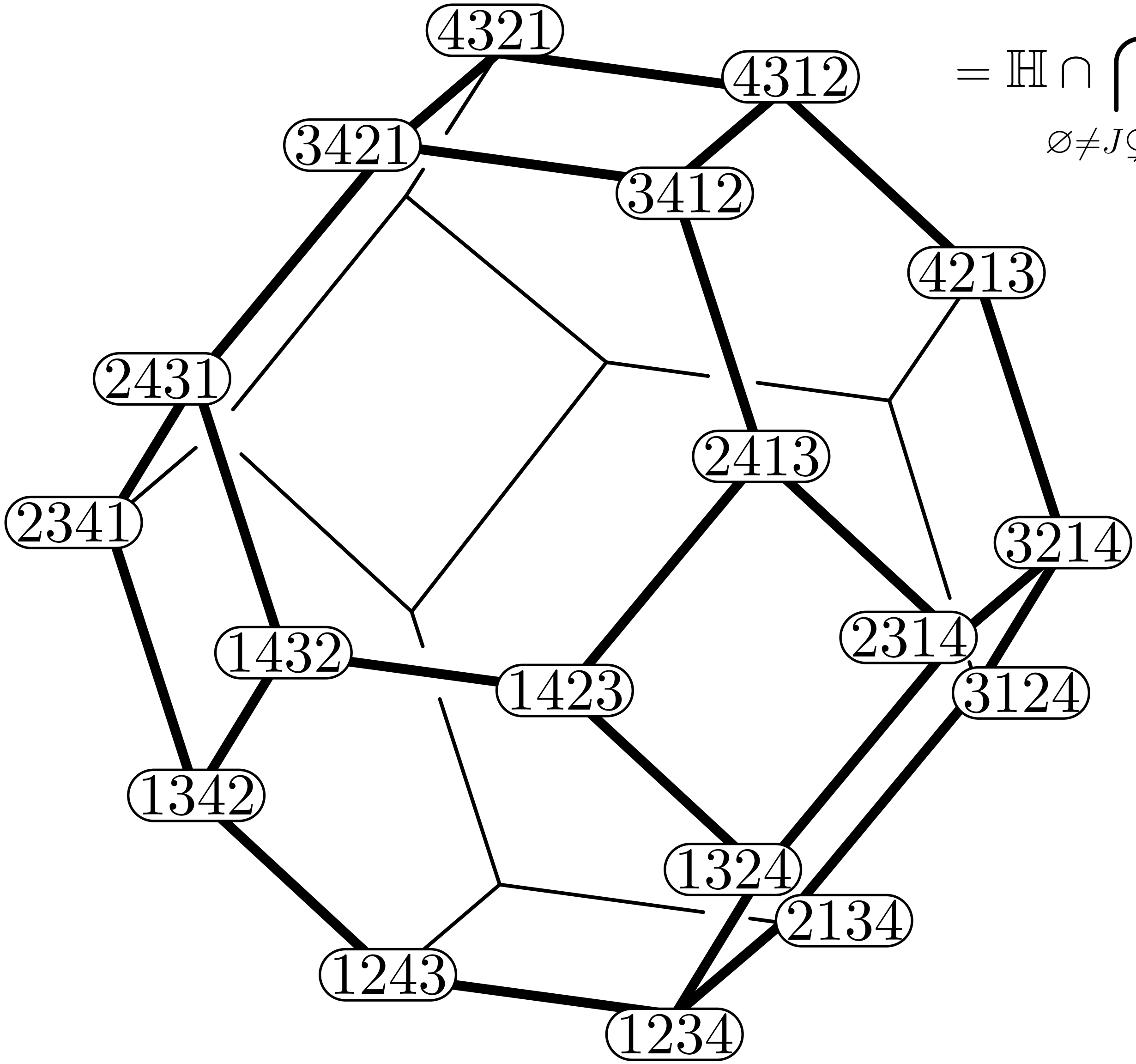

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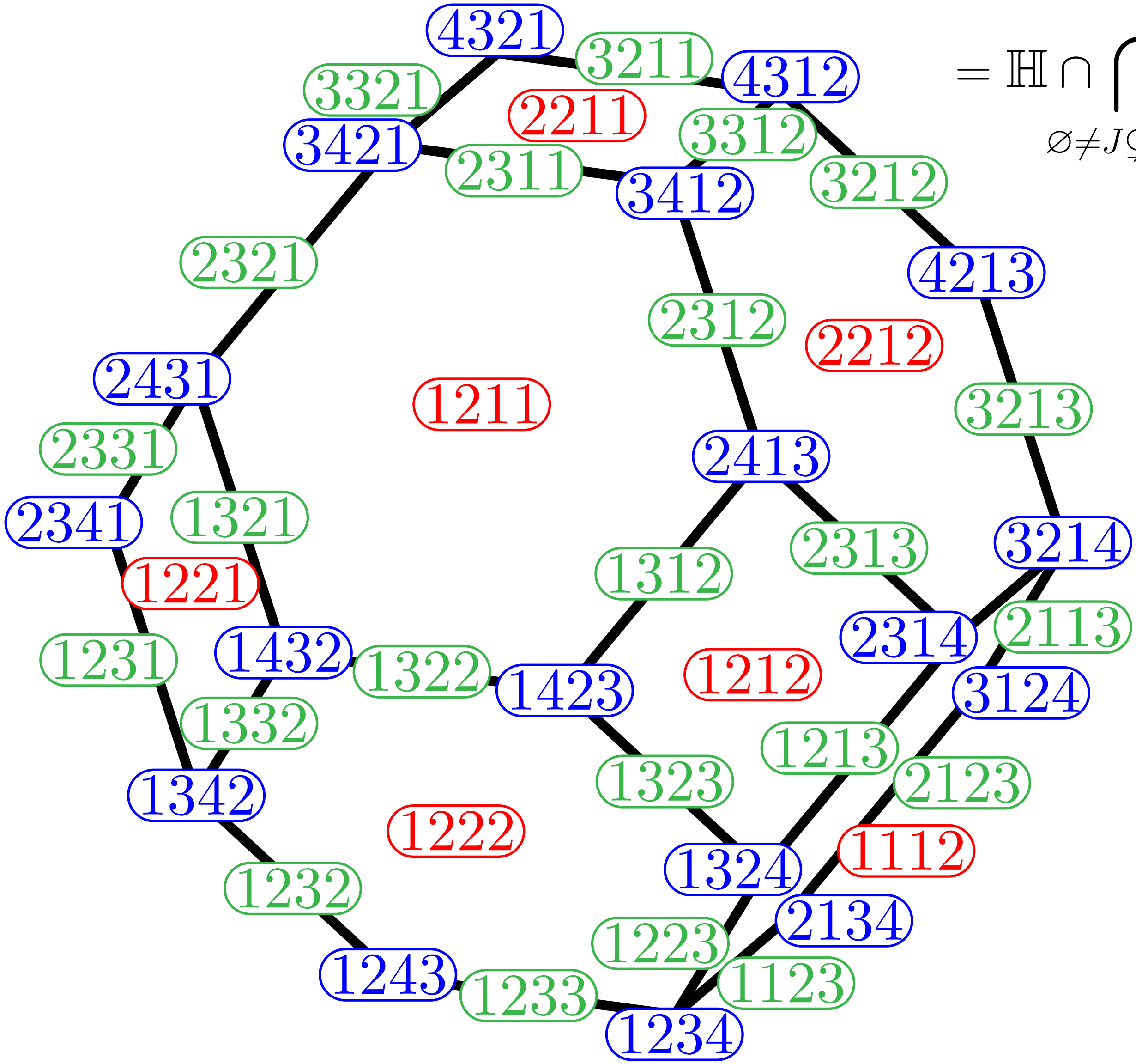
connections to

- weak order
- reduced expressions
- braid moves
- cosets of the symmetric group

PERMUTAHEDRON



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connections to

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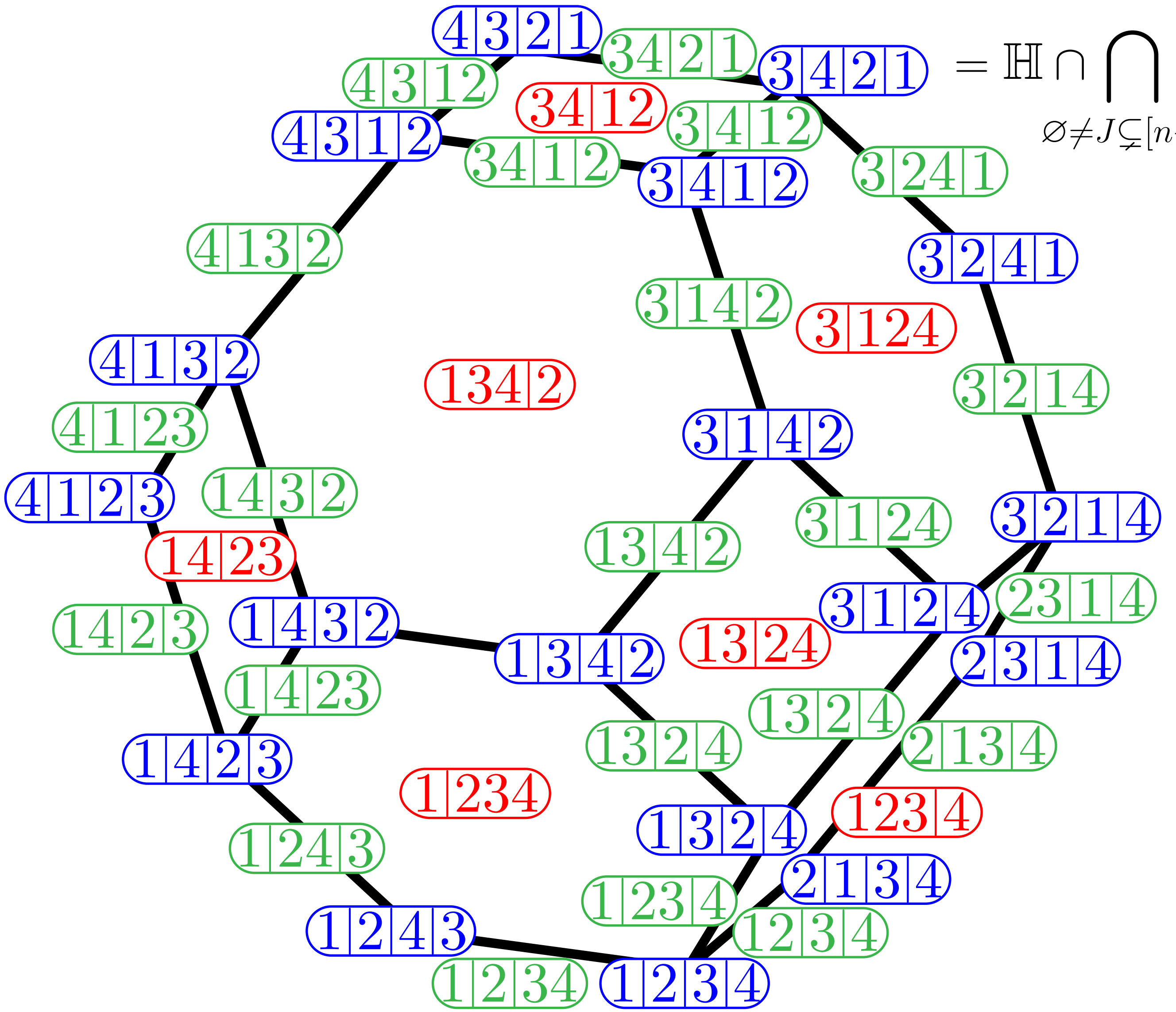
k-faces of $\mathsf{Perm}(n)$
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

PERMUTAHEDRON



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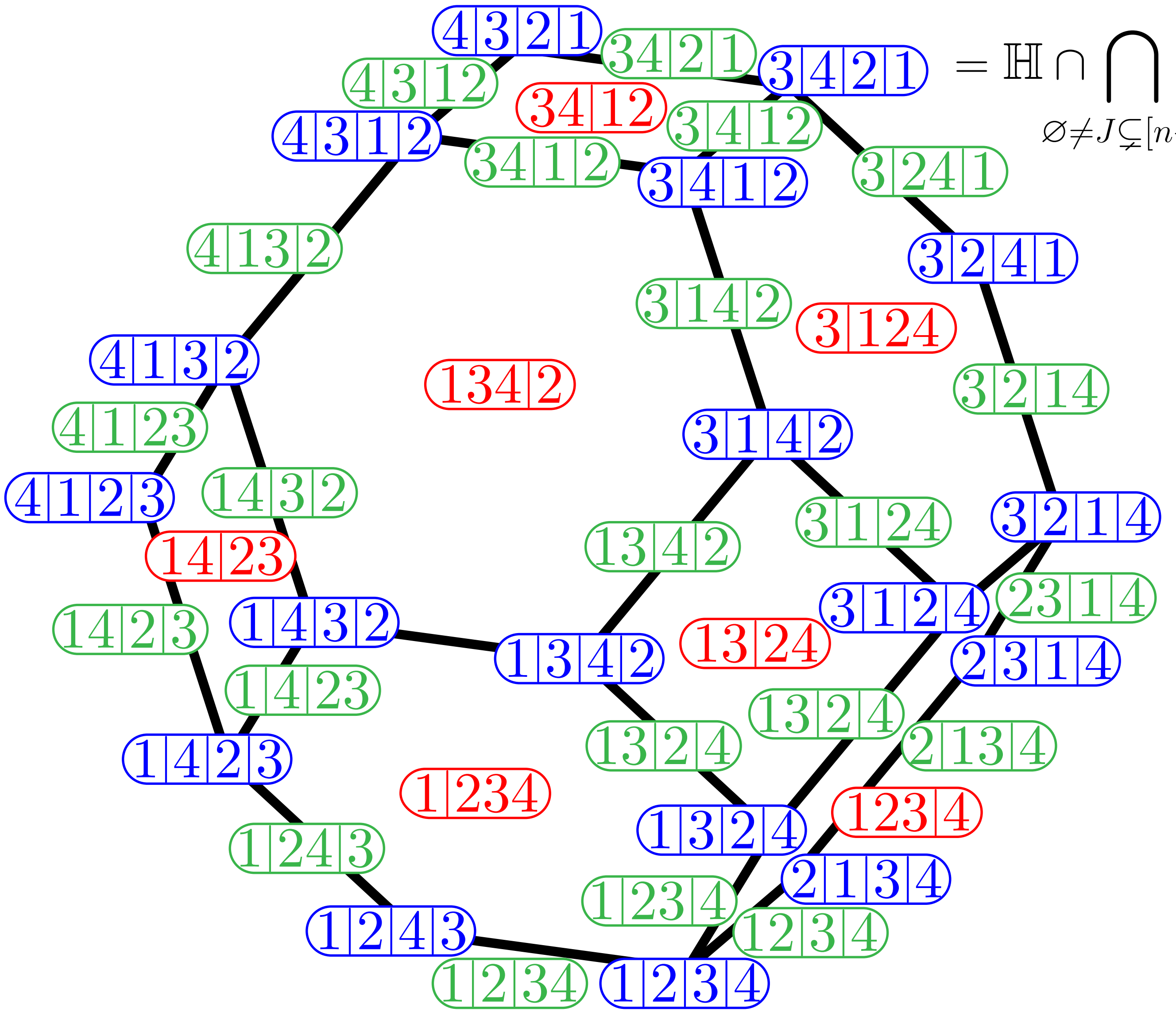
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

PERMUTAHEDRON



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connections to

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k-faces of Perm(n)

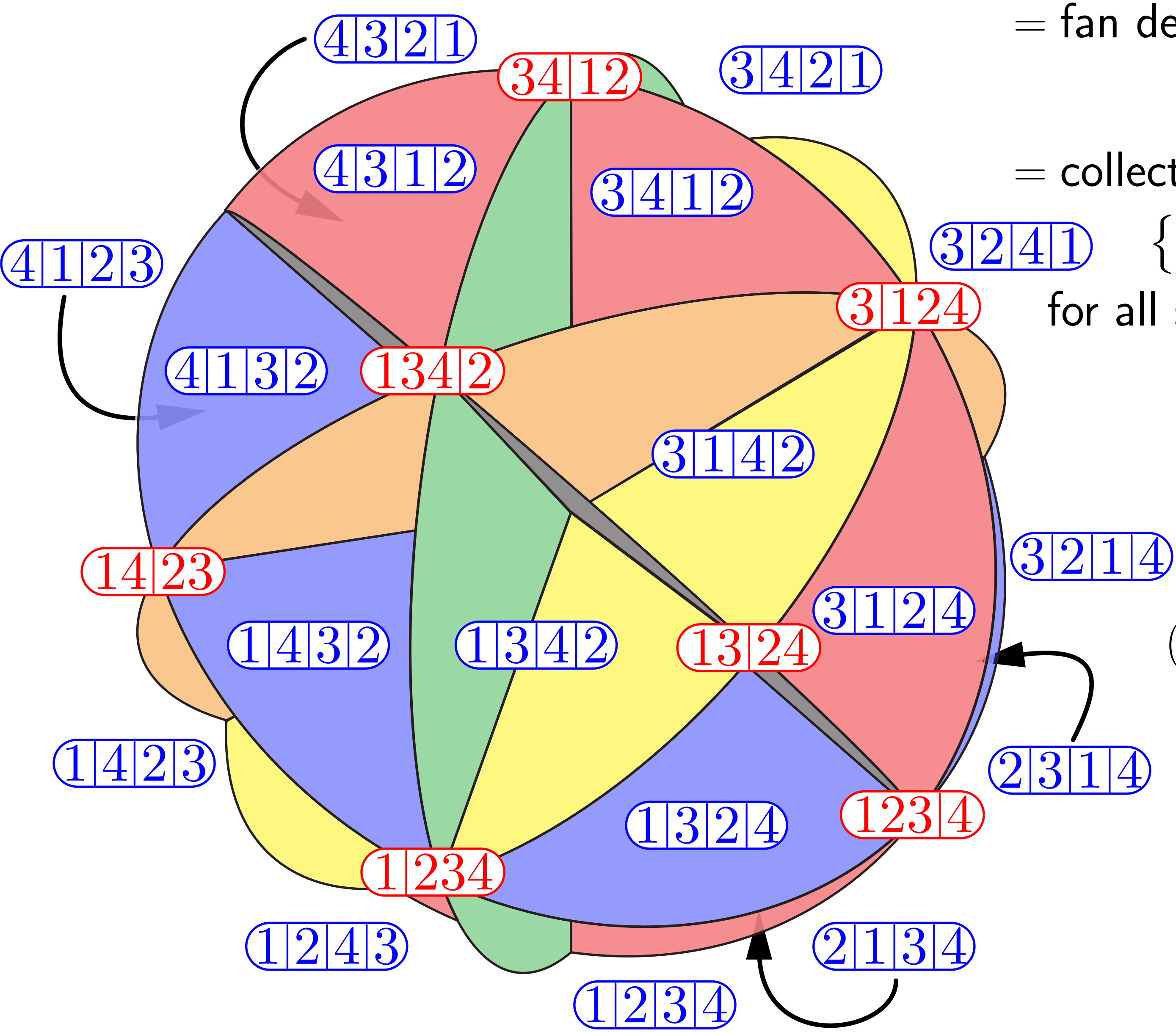
$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

COXETER ARRANGEMENT

Coxeter fan



= fan defined by the hyperplane arrangement

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i = x_j\right\}_{1 \le i < j \le n+1}$$

= collection of all cones

$$\left\{\mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; x_i < x_j \text{ if } \pi(i) < \pi(j)\right\}$$

for all surjections $\pi : [n+1] \to [n+1-k]$

$(n-k)$-dimensional cones

$\equiv$ surjections from $[n+1]$ to $[n+1-k]$

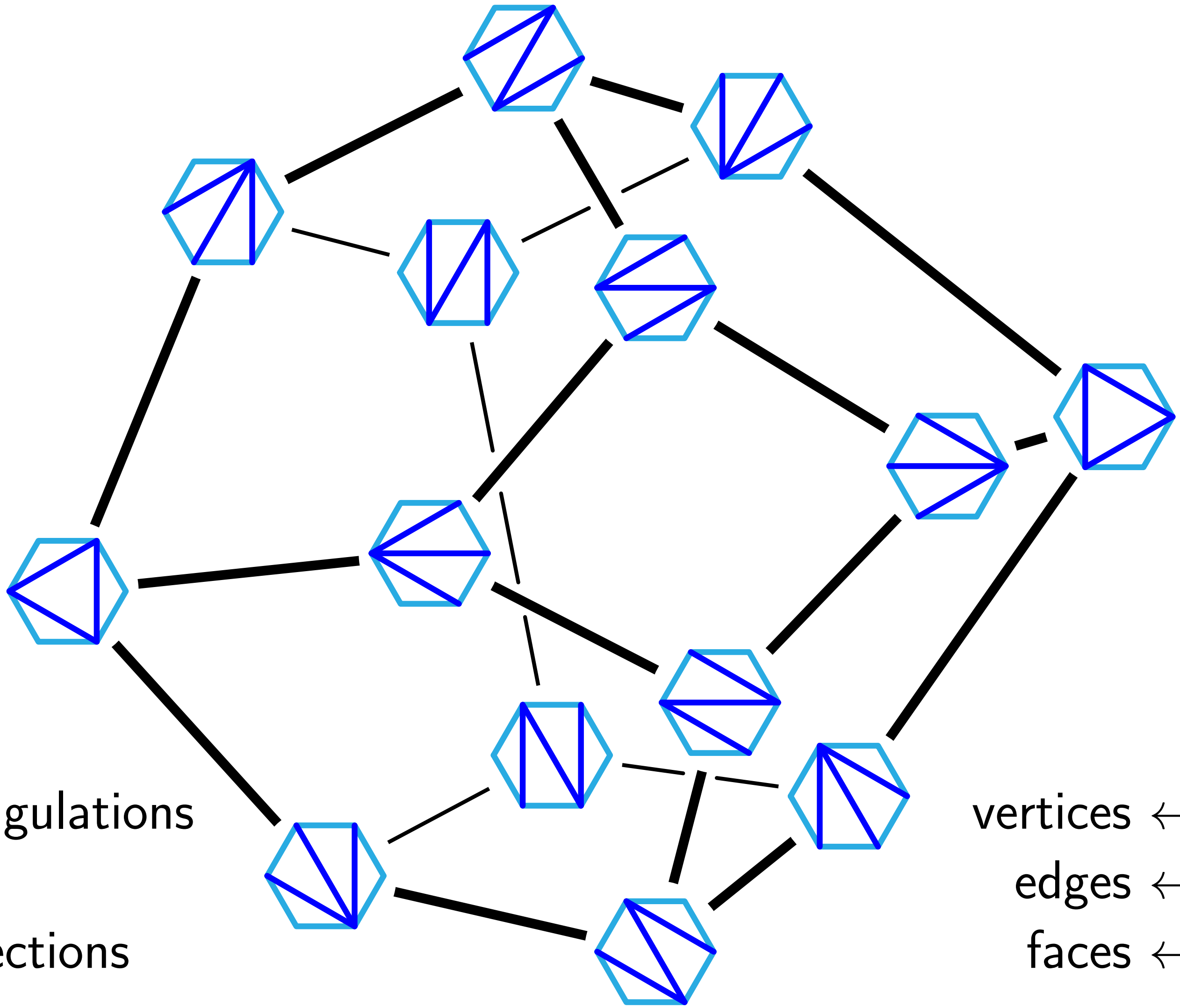
$\equiv$ ordered partitions of $[n+1]$ into $n+1-k$ parts

$\equiv$ collections of $n-k$ nested subsets of $[n+1]$

ASSOCIAHEDRA

ASSOCIAHEDRON

Associahedron $=$ polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion

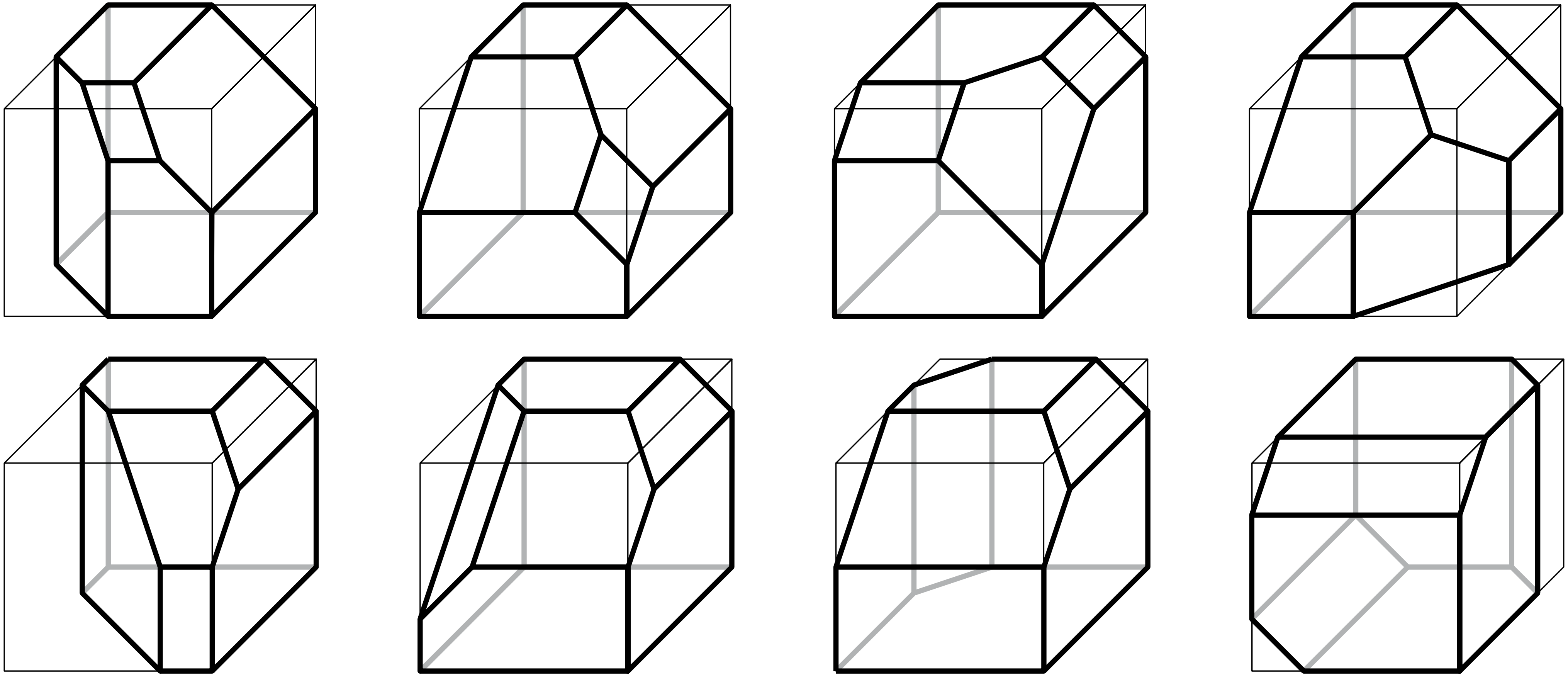


vertices $\leftrightarrow$ triangulations
edges $\leftrightarrow$ flips
faces $\leftrightarrow$ dissections

vertices $\leftrightarrow$ binary trees
edges $\leftrightarrow$ rotations
faces $\leftrightarrow$ Schröder trees

VARIOUS ASSOCIAHEDRA

Associahedron = polytope whose face lattice is isomorphic to the lattice of crossing-free sets of internal diagonals of a convex $(n+3)$-gon, ordered by reverse inclusion



(Pictures by Ceballos-Santos-Ziegler)

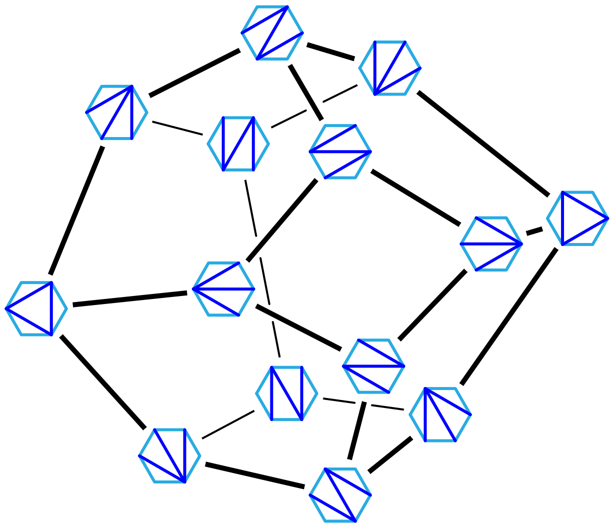
Tamari ('51) — Stasheff ('63) — Haimann ('84) — Lee ('89) —

... — Gel'fand-Kapranov-Zelevinski ('94) — ... — Chapoton-Fomin-Zelevinsky ('02) — ... — Loday ('04) — ...

— Ceballos-Santos-Ziegler ('11)

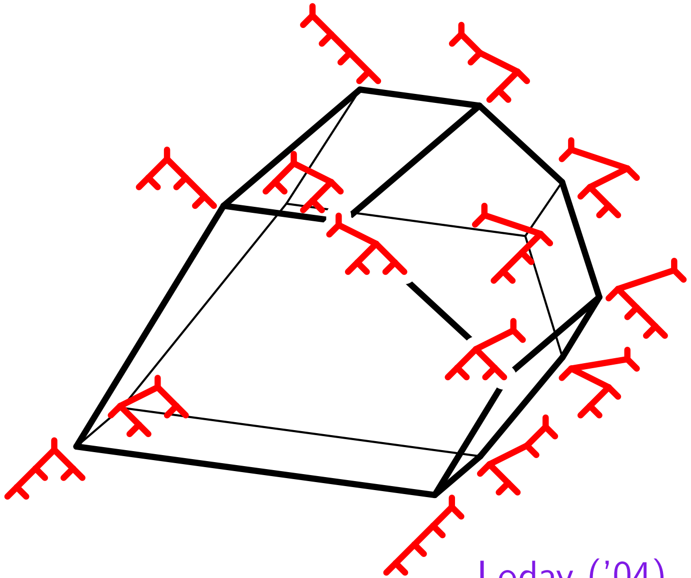
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

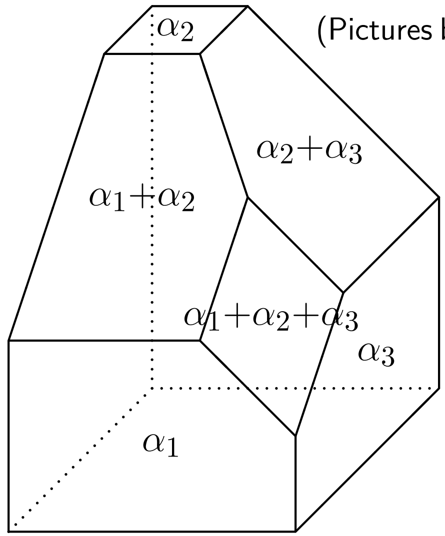
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

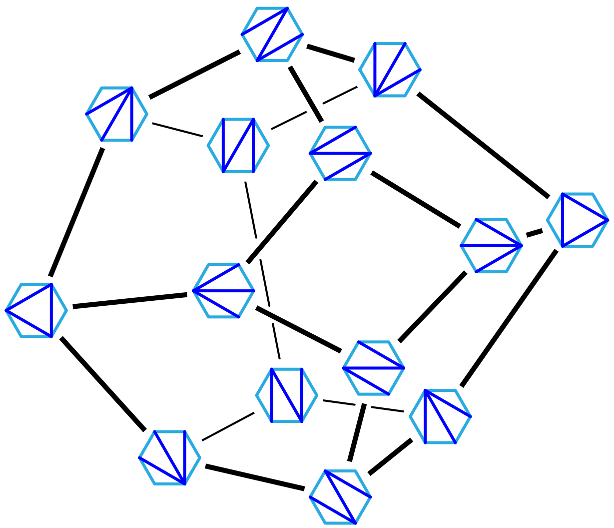
(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
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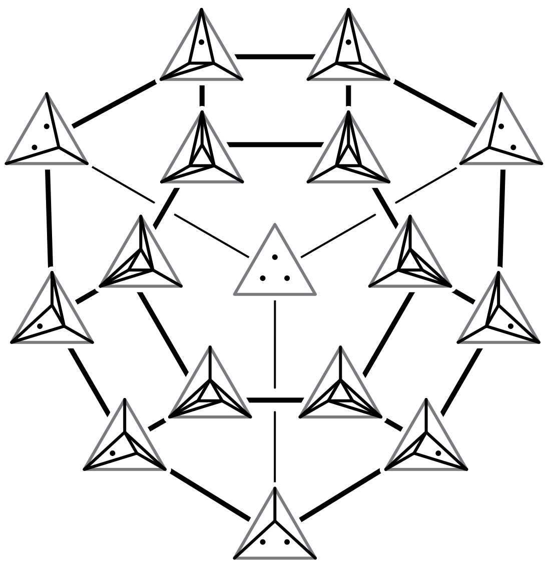
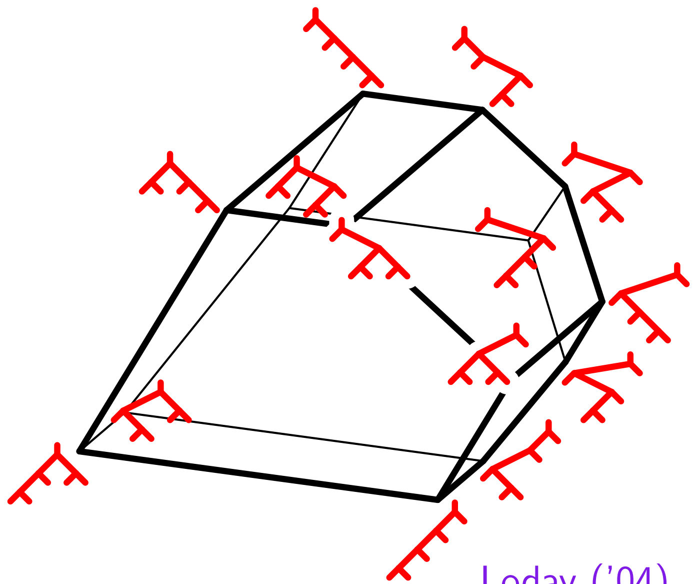


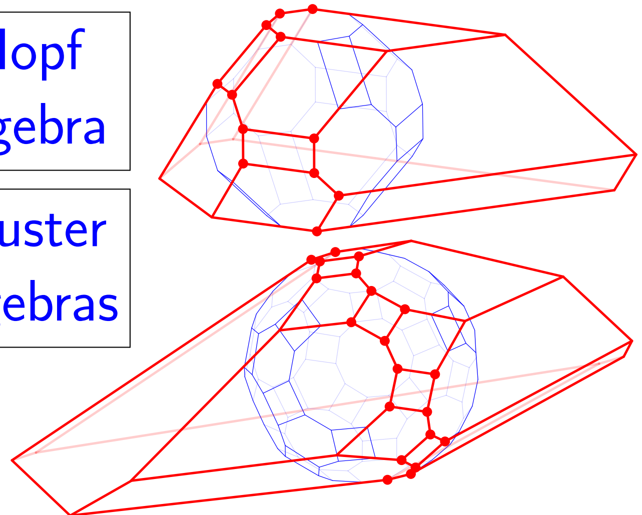
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

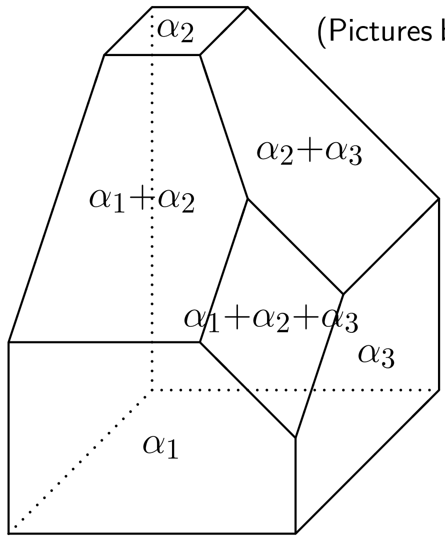
Hopf algebra

Cluster algebras



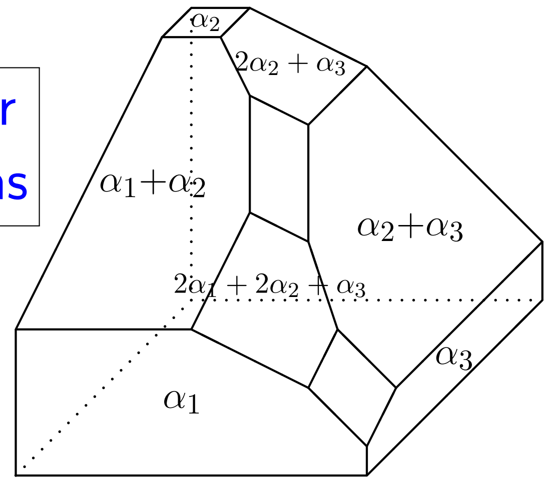
CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

(Pictures by CFZ)



Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

Cluster algebras

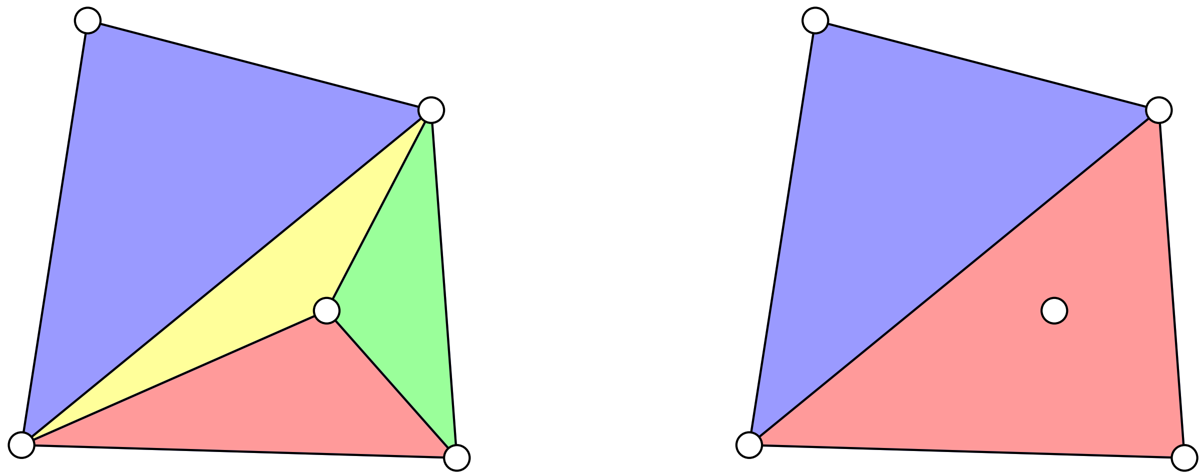


SECONDARY POLYTOPES

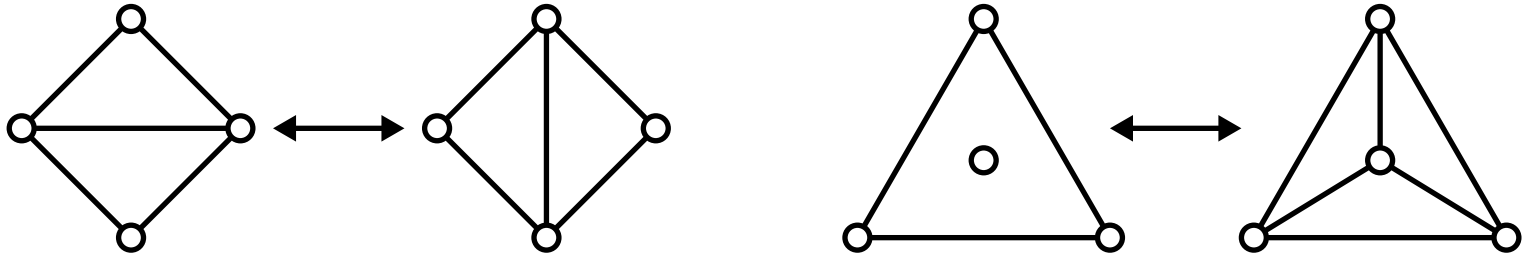
TRIANGULATIONS AND SUBDIVISIONS

triangulation of $\mathbf{P} \subset \mathbb{R}^d$ = collection of triangles with corners in $\mathbf{P}$ such that

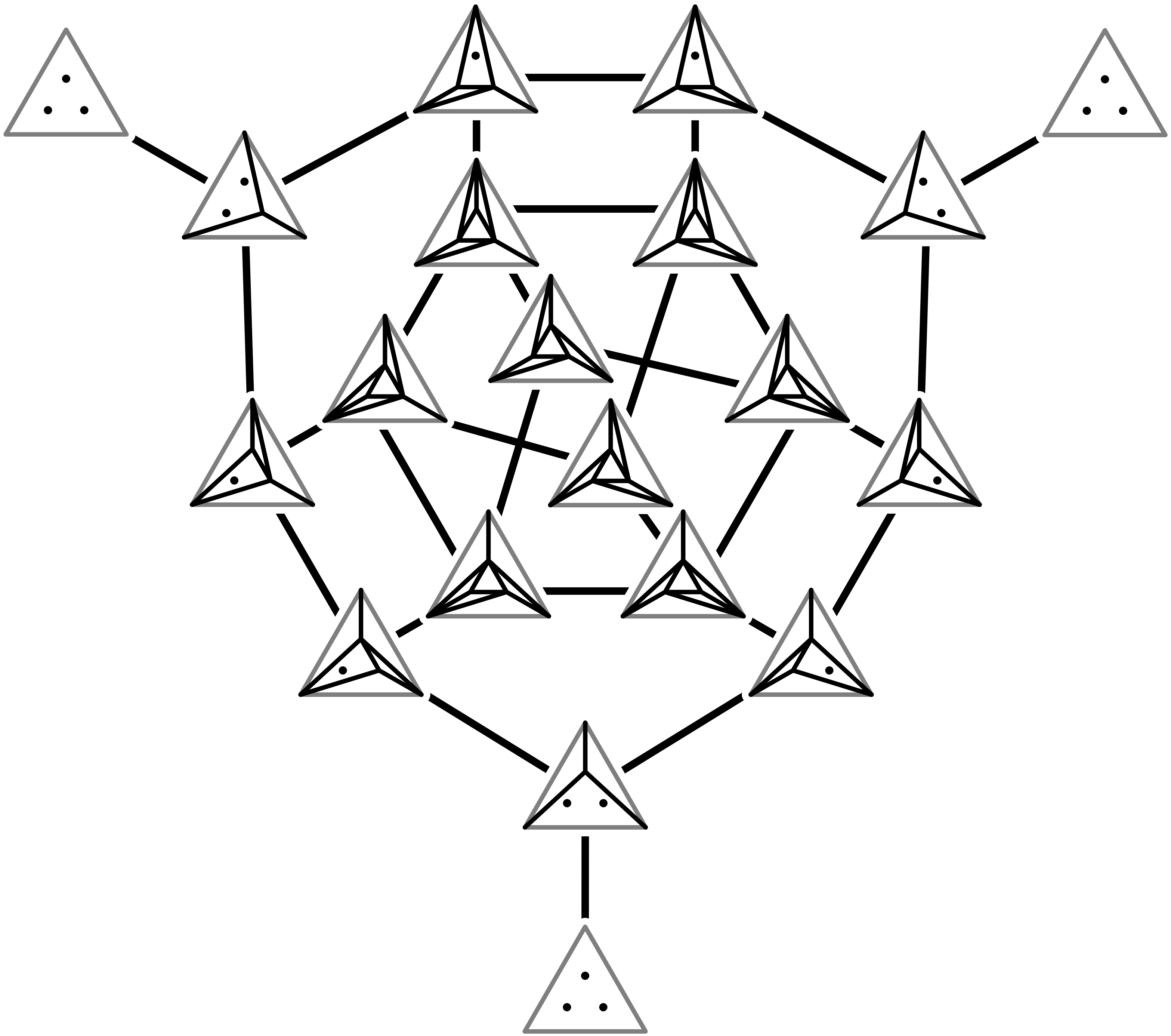
- covering property: their union cover the convex hull of $\mathbf{P}$,
- intersection property: any two triangles intersect in a proper face.



flip =



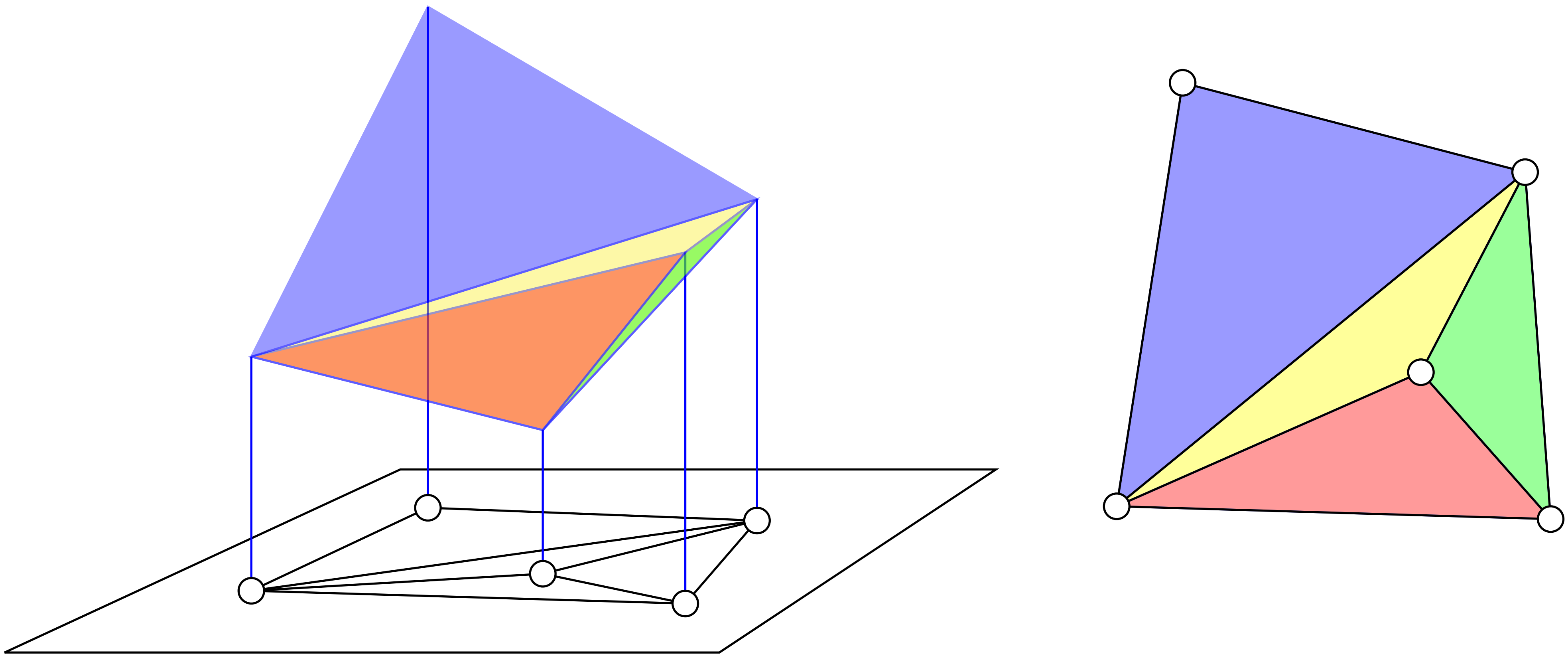
FLIP GRAPH



REGULAR SUBDIVISIONS

$\mathbf{P}$ point set in $\mathbb{R}^d$

$\omega : \mathbf{P} \to \mathbb{R}$ height function



$\mathrm{Sub}(\mathbf{P}, \omega)$ = projection of the lower convex hull of the point set $\{(\mathbf{p}, \omega(\mathbf{p})) \mid \mathbf{p} \in \mathbf{P}\}$

regular subdivision = subdivision S such that $\exists\, \omega : \mathbf{P} \to \mathbb{R}^d$ for which $S = \mathrm{Sub}(\mathbf{P}, \omega)$

REGULAR SUBDIVISIONS

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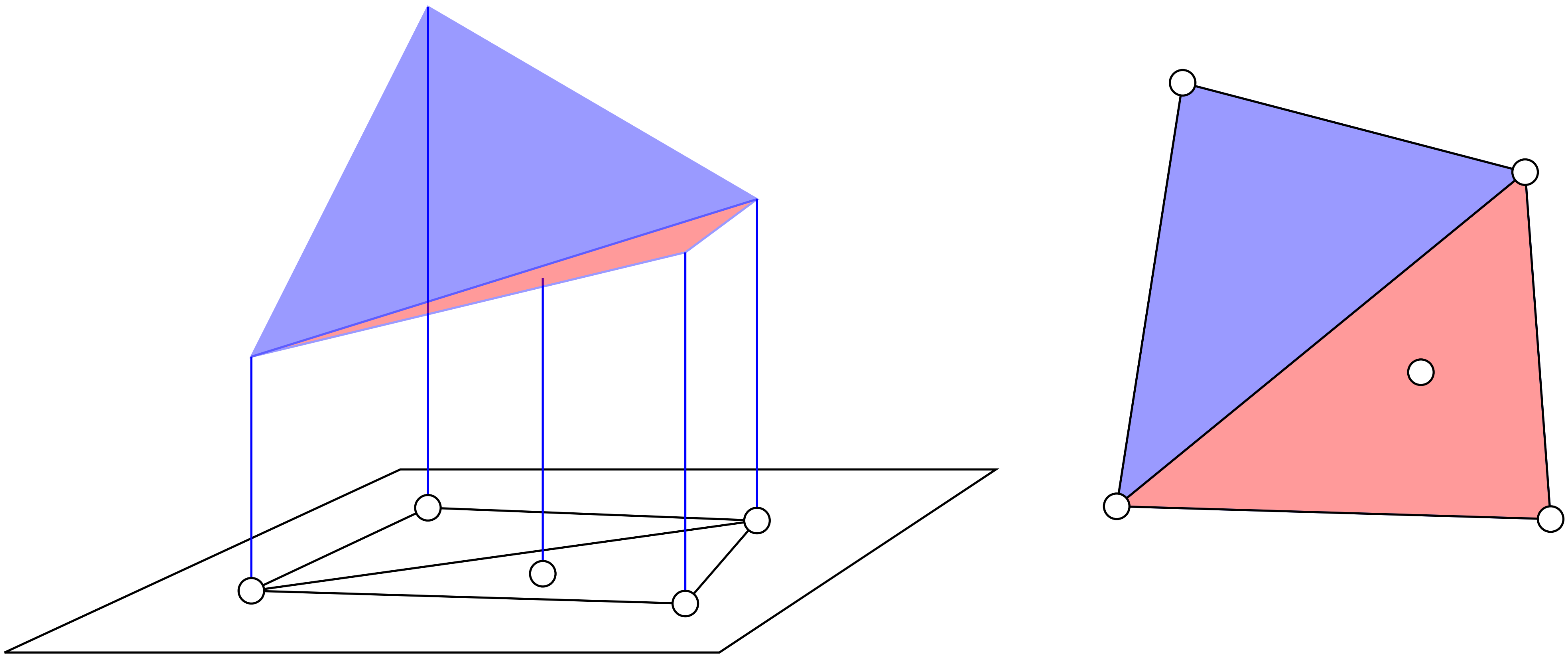

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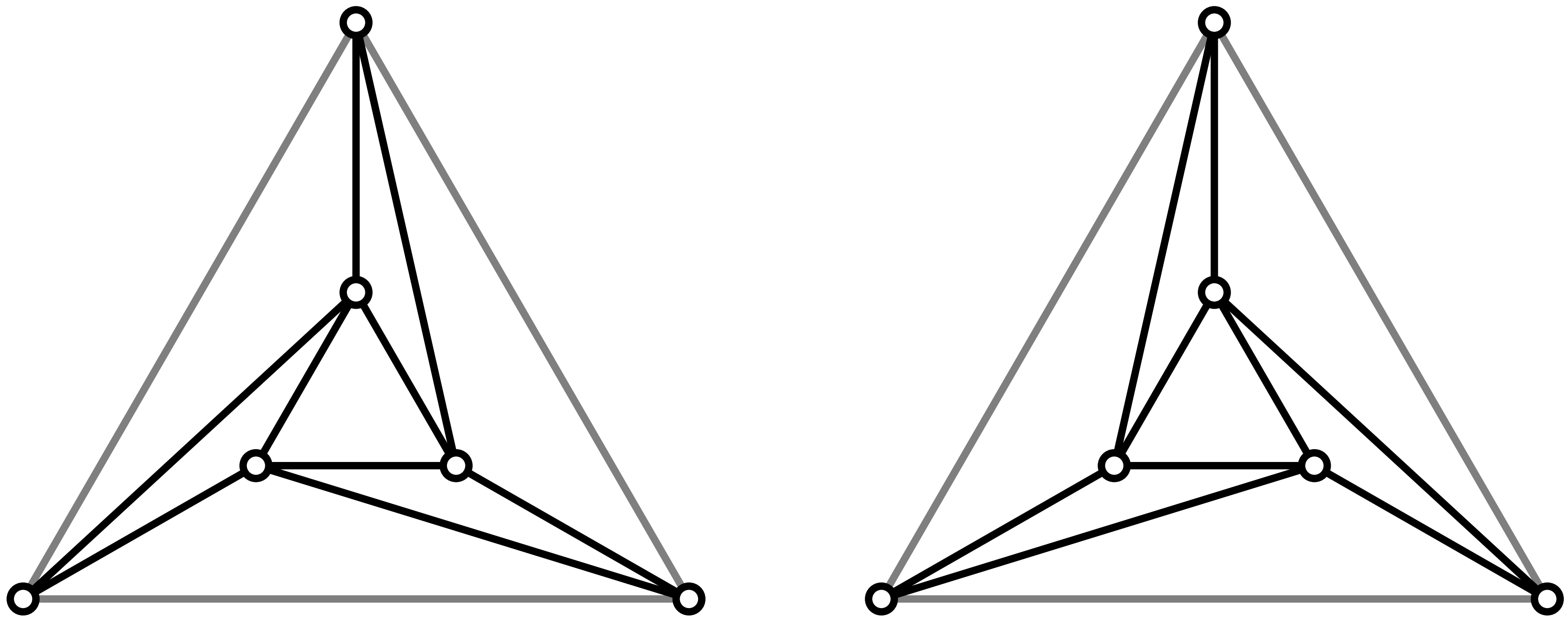
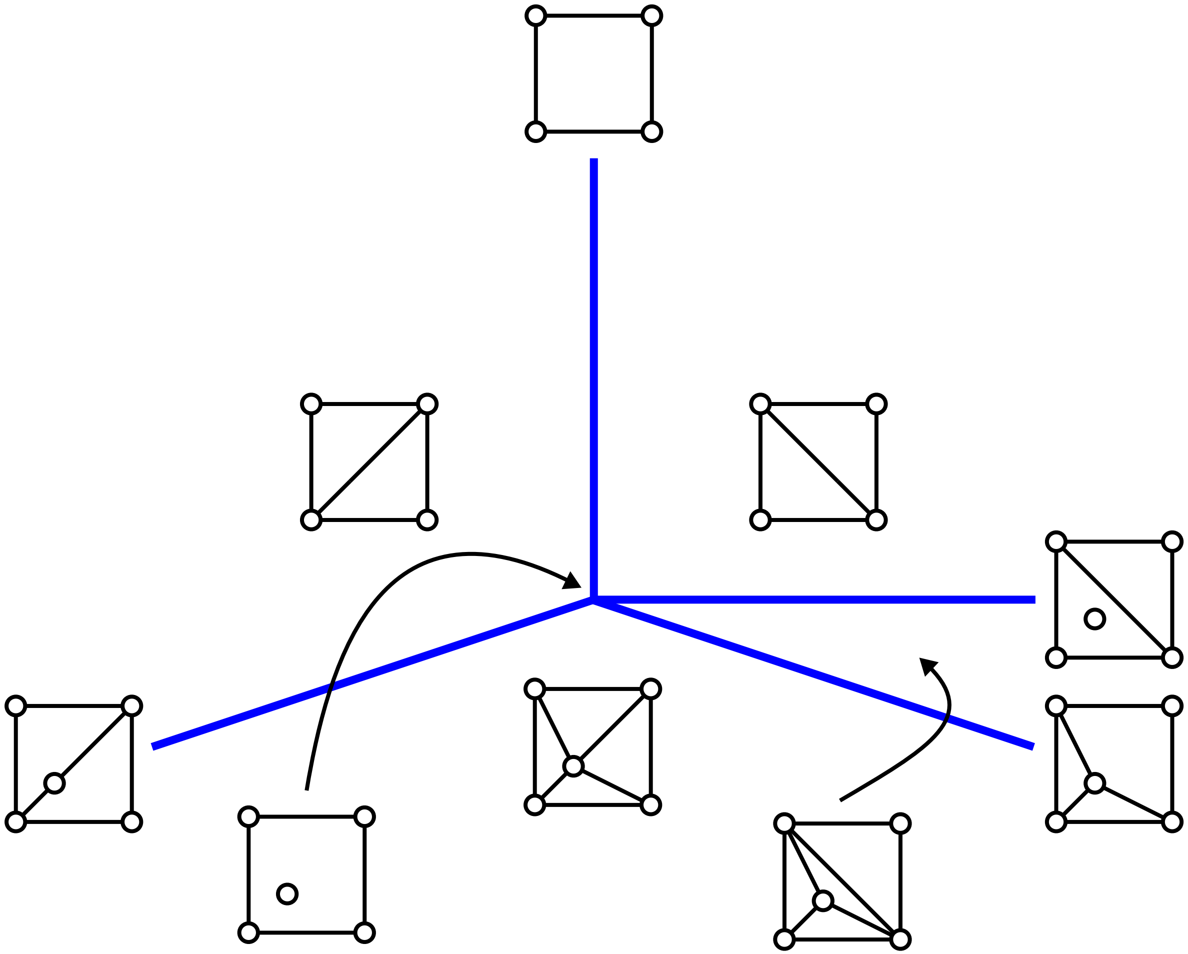
NON-REGULAR TRIANGULATIONS



All triangulations of a convex polygon are regular

SECONDARY FAN

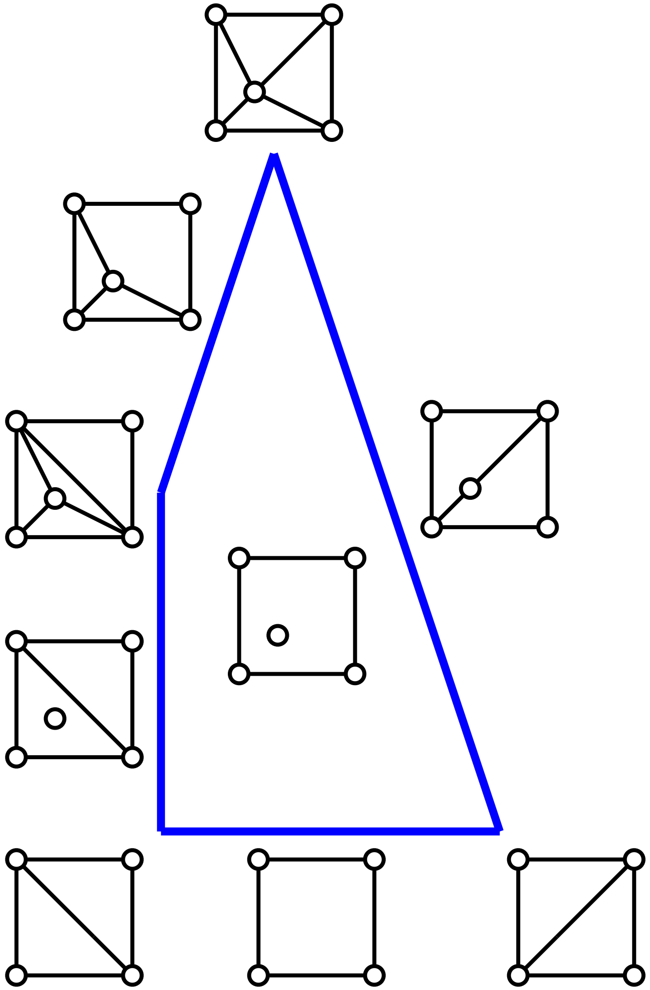
secondary cone of a subdivision S of $\mathbf{P} = C(S) = \left\{\omega \in \mathbb{R}^{\mathbf{P}} \;\middle|\; S \text{ refines } S(\mathbf{P}, \omega)\right\}$
secondary fan of $\mathbf{P} = \{C(S) \mid S \text{ subdivision of } \mathbf{P}\}$



SECONDARY POLYTOPE

volume vector of a triangulation T of $\mathbf{P}$ = $\Phi(T) = \left(\sum_{\mathbf{p} \in \Delta \in T} \mathrm{vol}(\Delta) \right)_{\mathbf{p} \in \mathbf{P}} \in \mathbb{R}^{\mathbf{P}}$

secondary polytope of $\mathbf{P}$ = convex hull of $\{\Phi(T) \mid T \text{ triangulation of } \mathbf{P}\}$

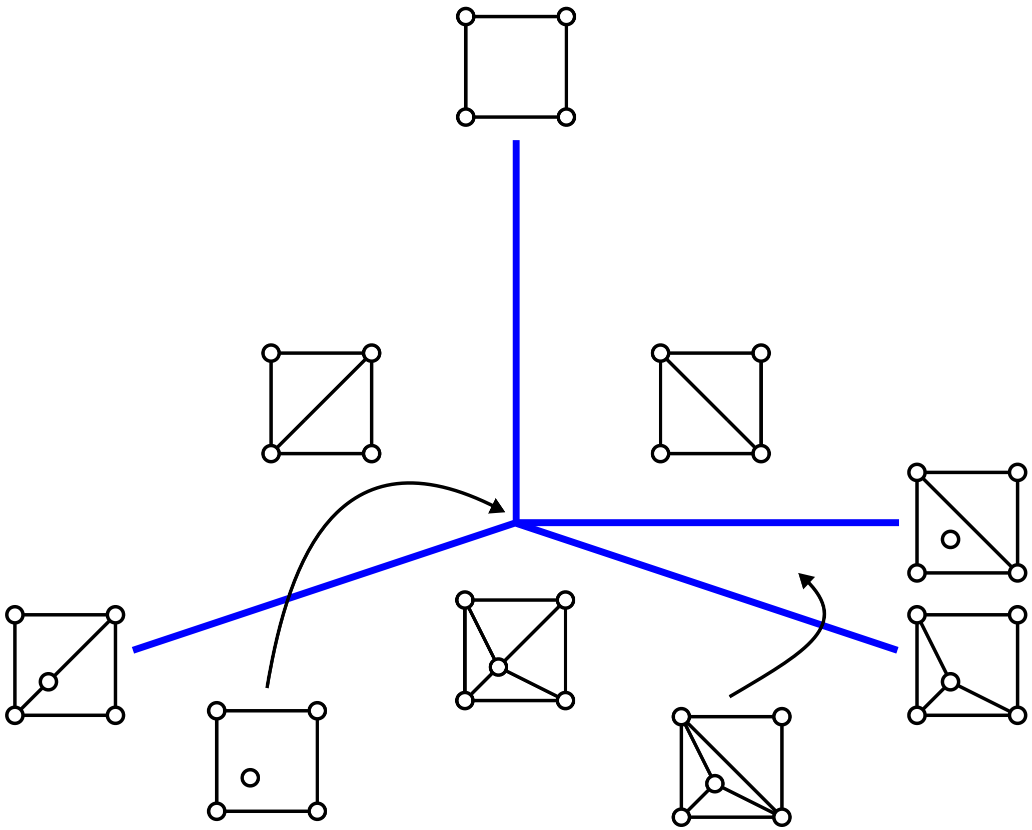


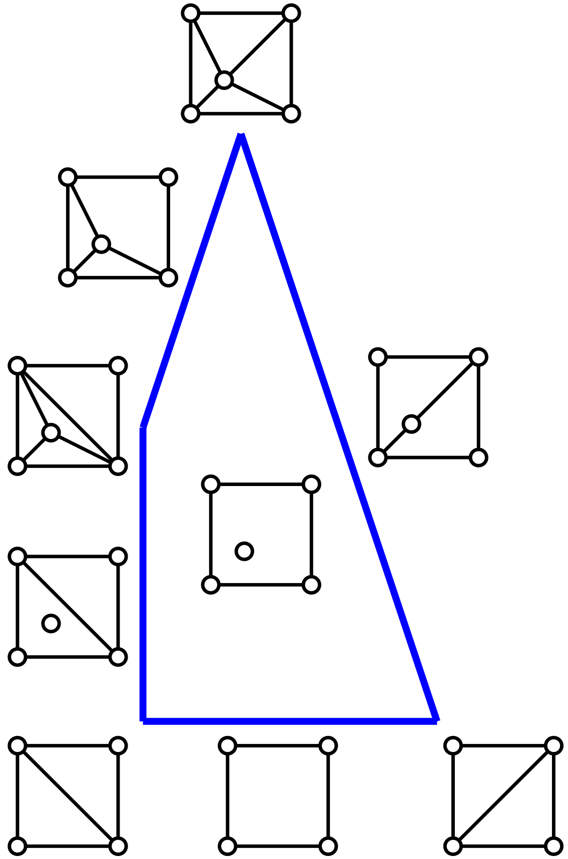
SECONDARY POLYTOPE

THM. For a point set $\mathbf{P} \subset \mathbb{R}^{\mathbf{P}}$:

1. The secondary polytope of $\mathbf{P}$ has dimension $|\mathbf{P}| - d - 1$.
2. The secondary fan of $\mathbf{P}$ is the inner normal fan of the secondary polytope of $\mathbf{P}$.
3. The face lattice of the secondary polytope of $\mathbf{P}$ is isom. to the refinement poset of regular subdivisions of $\mathbf{P}$.

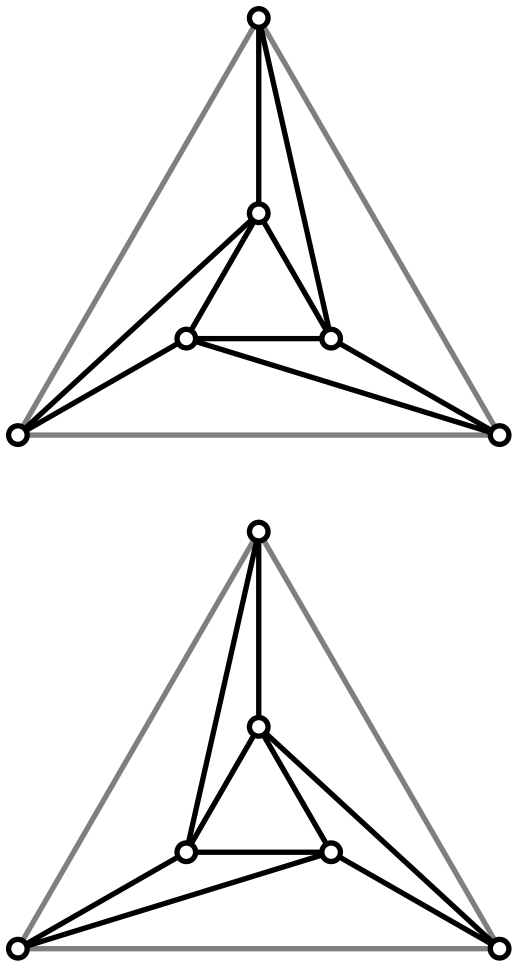
Gelfand-Kapranov-Zelevinsky, Discriminants, resultants, and multidimensional determinants ('94)

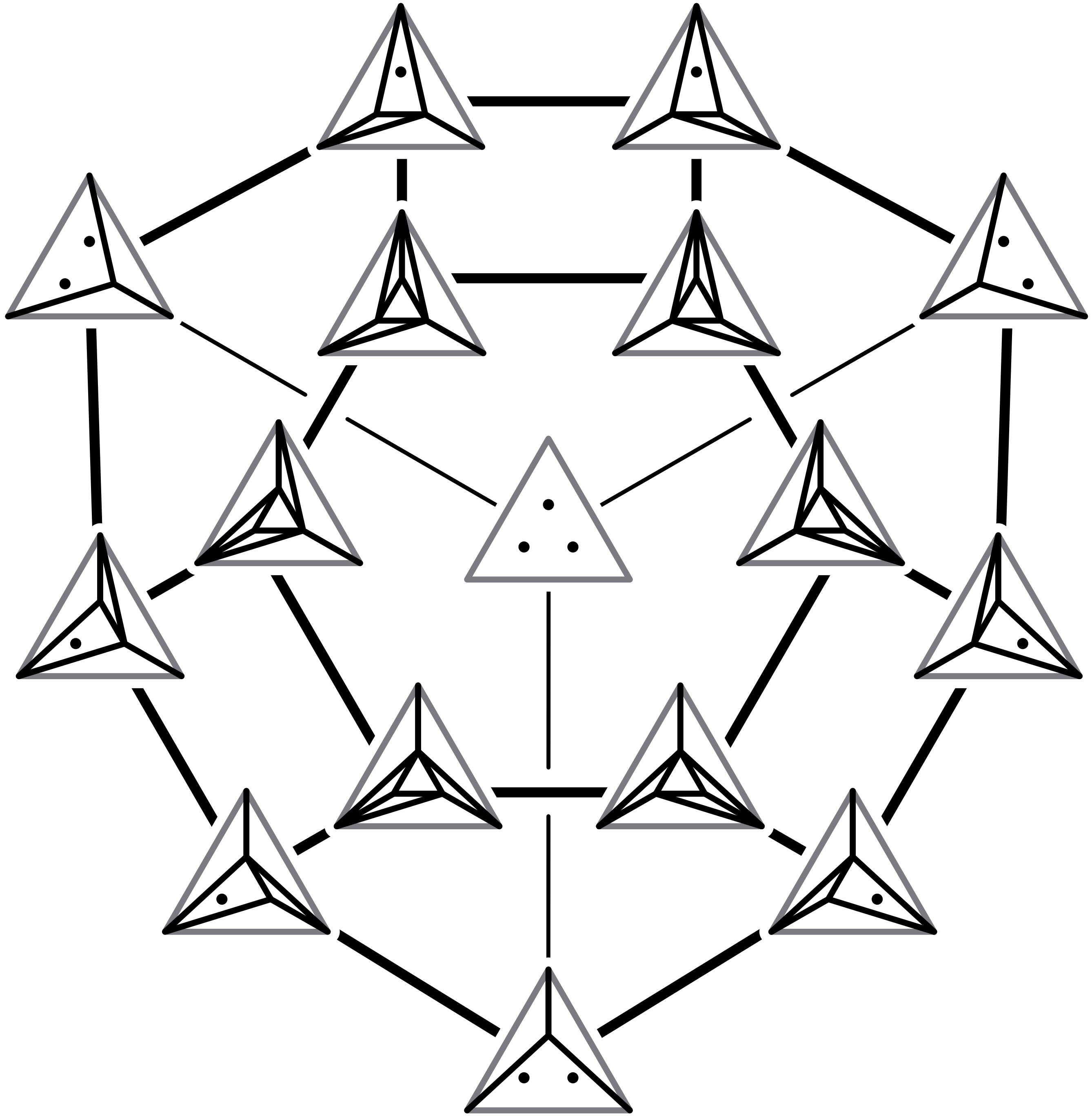




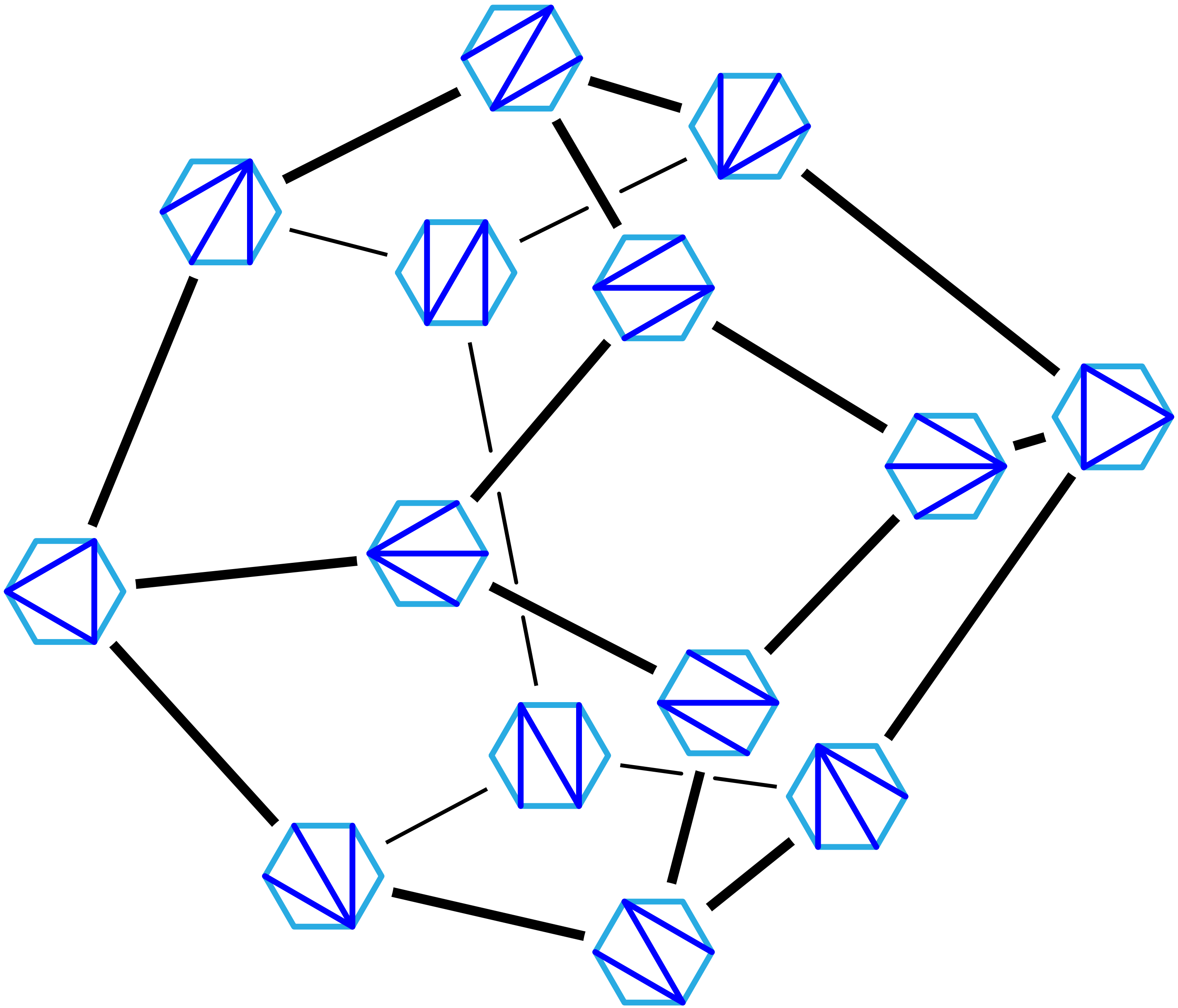
SECONDARY POLYTOPE

Non-regular triangulations and subdivisions are invisible





Secondary polytope of a convex polygon = associahedron

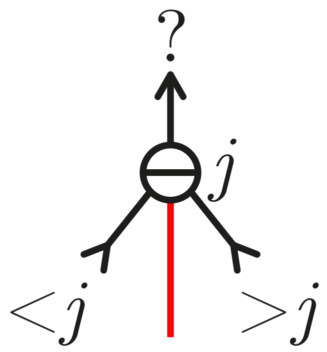


LODAY'S ASSOCIAHEDRON

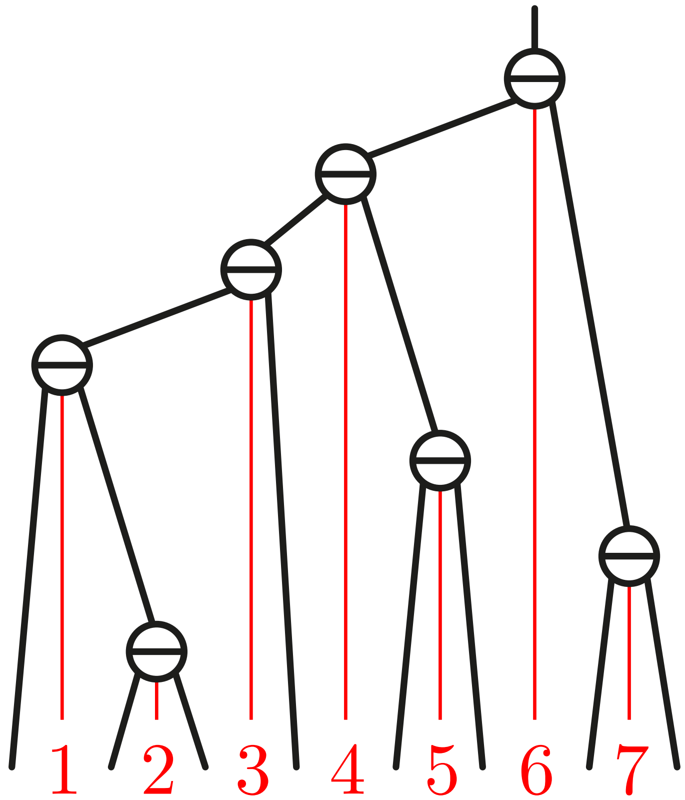
BINARY TREES

T binary tree

Infix search labeling $=$ labeling with $[n]$ with the following local rule

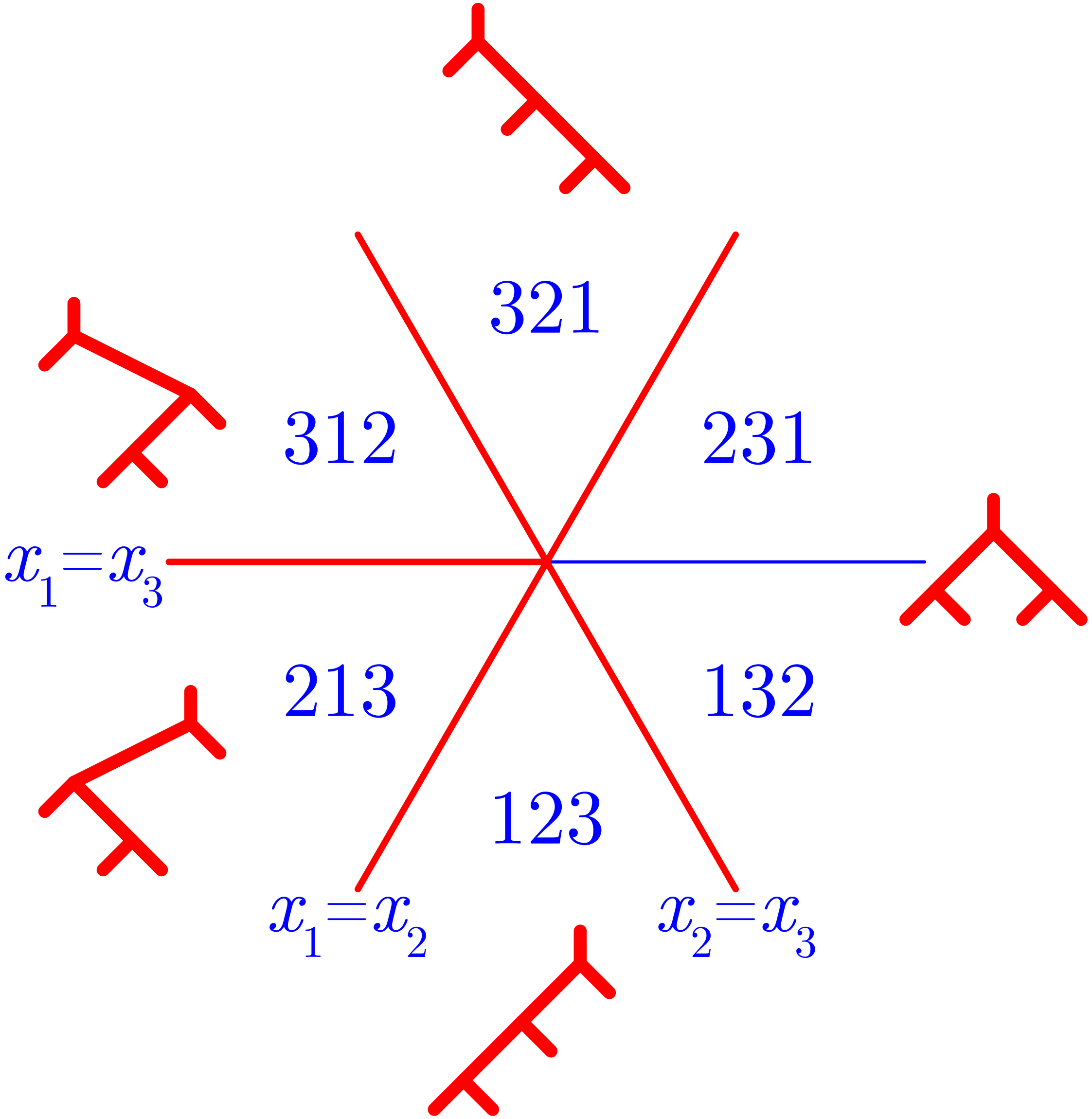


Rule for binary search trees



SYLVESTER FAN

cone of a binary tree $T = C(T) = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j$ for each edge $i \rightarrow j$ in $T\}$
sylvester fan $= \{C(T) \mid T$ binary tree on n nodes$\}$

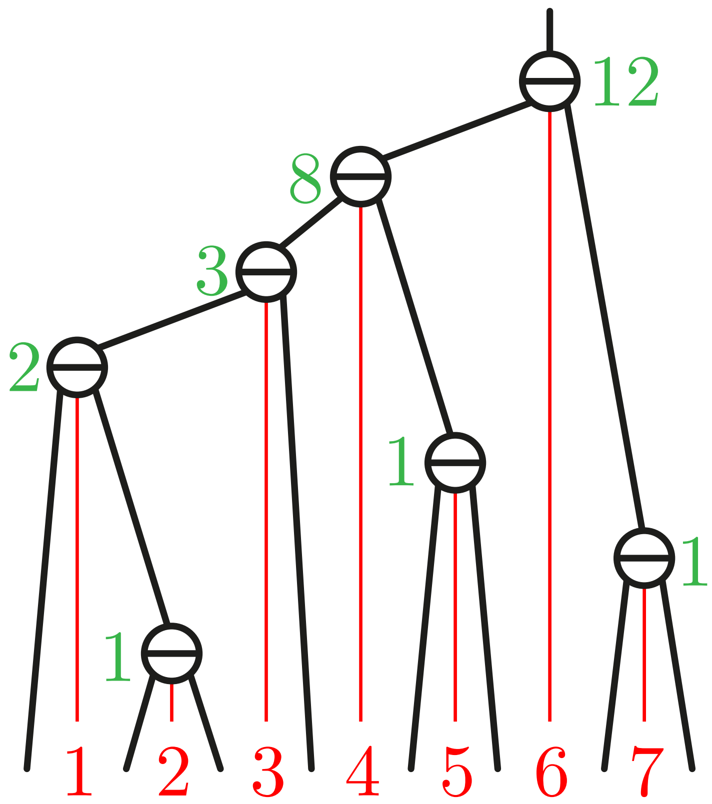


LODAY'S ASSOCIAHEDRON

$$\mathsf{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

Loday, Realization of the Stasheff polytope ('04)



LODAY'S ASSOCIAHEDRON

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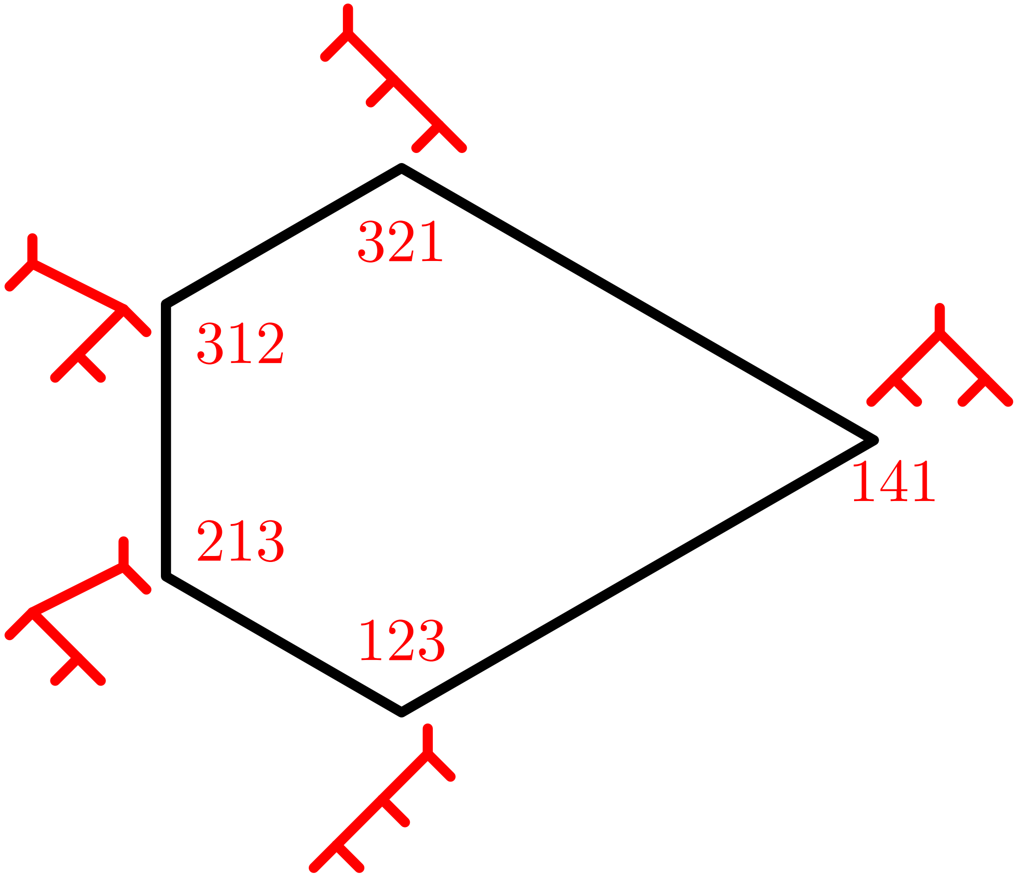
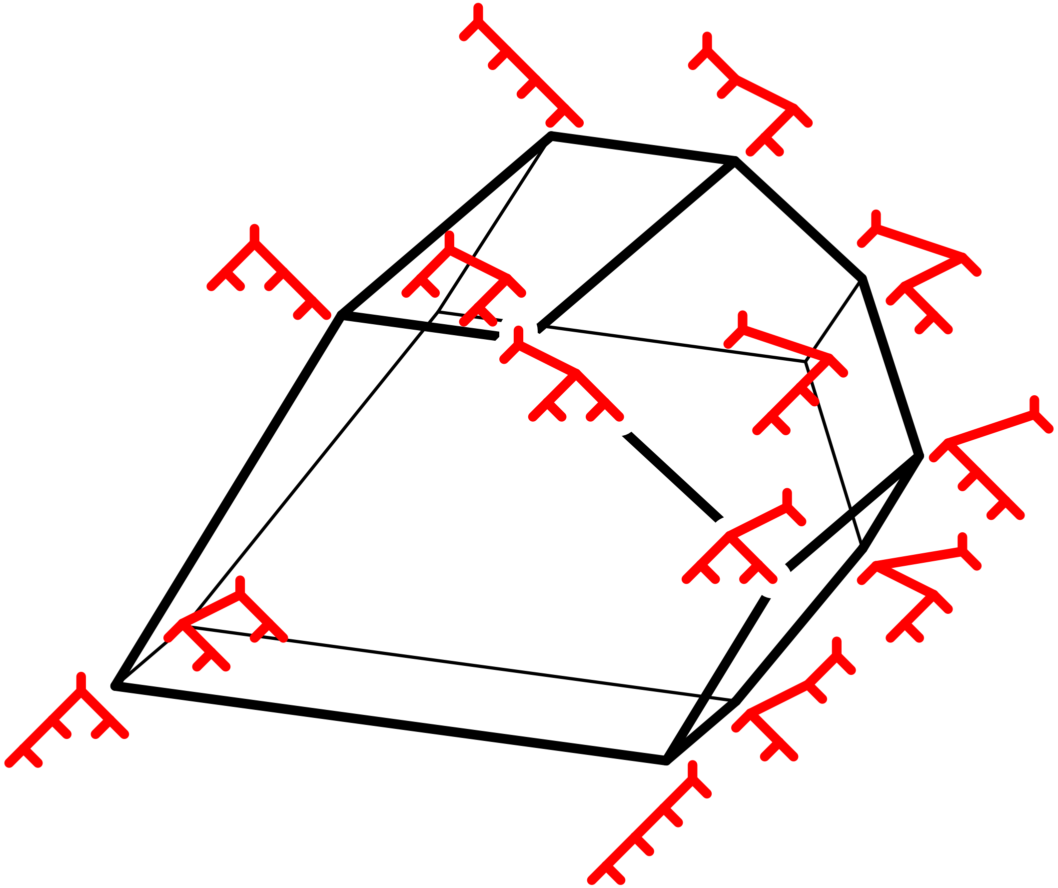
Loday, Realization of the Stasheff polytope ('04)



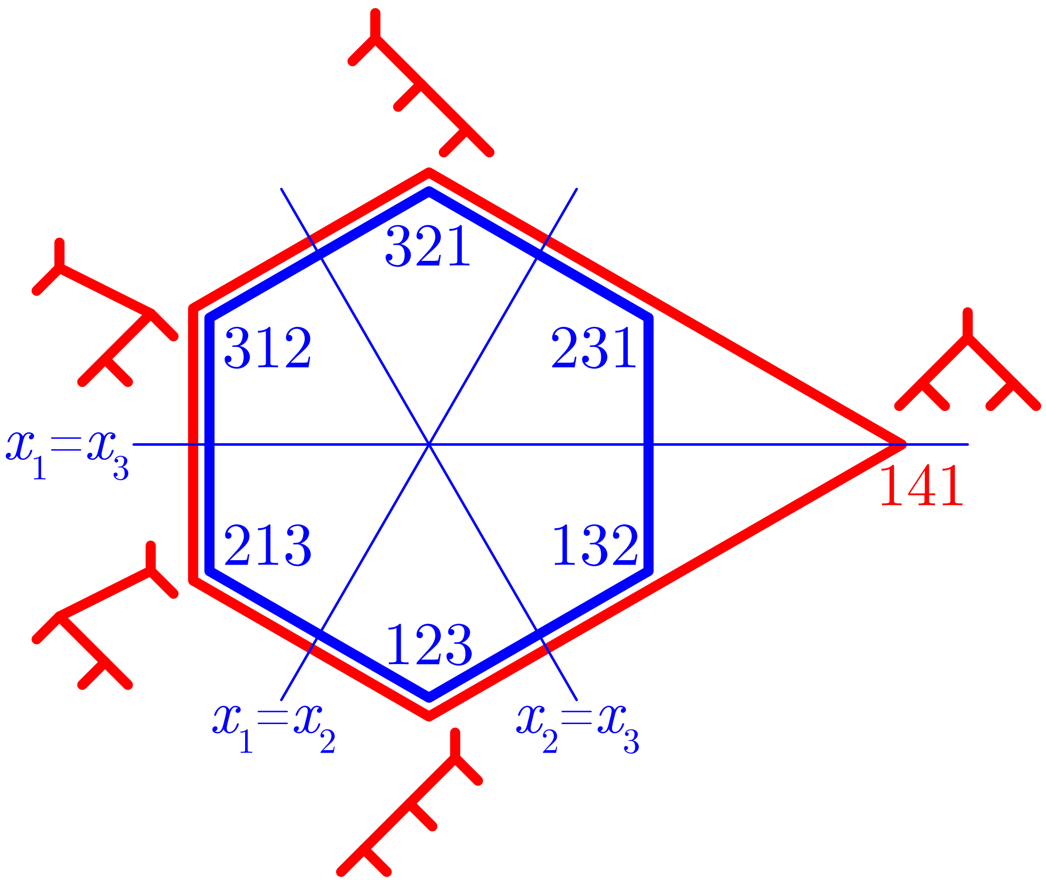


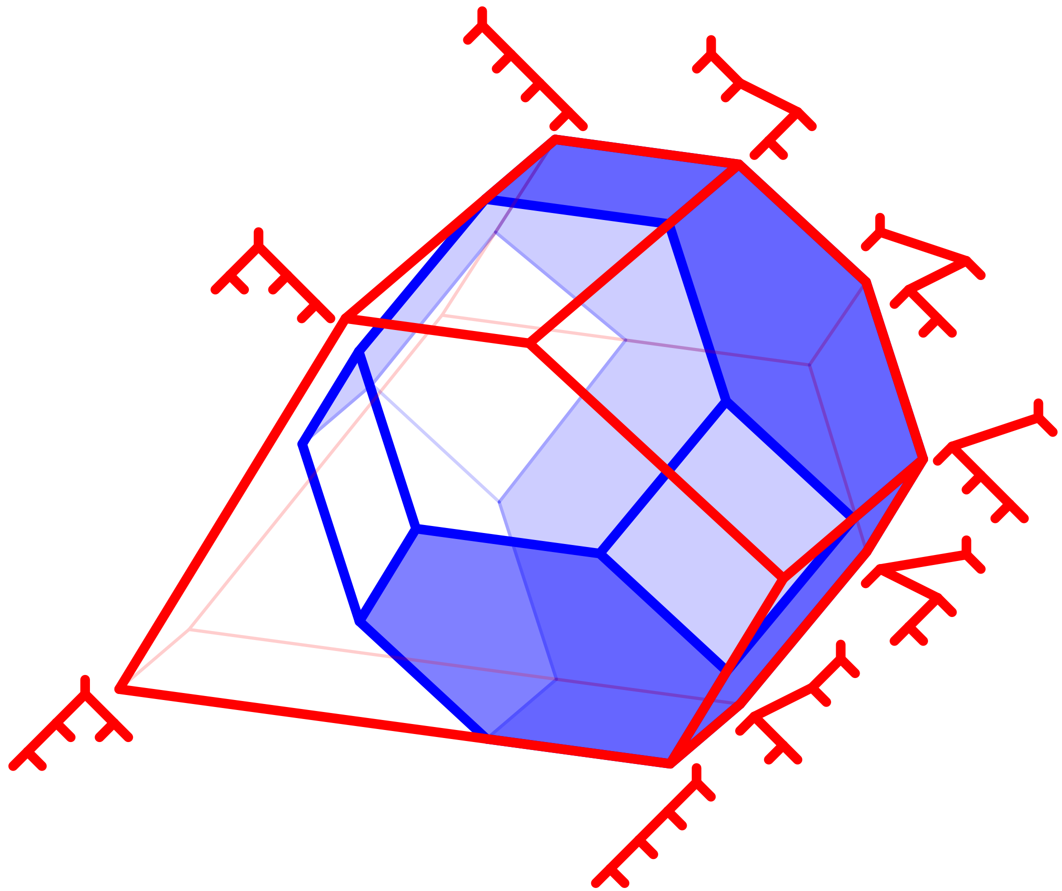
LODAY'S ASSOCIAHEDRON

$$\mathsf{Asso}(n) := \operatorname{conv}\{\mathbf{L}(\mathrm{T}) \mid \mathrm{T} \text{ binary tree}\} = \mathbb{H} \cap \bigcap_{1 \le i \le j \le n+1} \mathbf{H}^{\ge}(i,j)$$

$$\mathbf{L}(\mathrm{T}) := \big[\ell(\mathrm{T},i) \cdot r(\mathrm{T},i)\big]_{i \in [n+1]} \qquad \mathbf{H}^{\ge}(i,j) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i \le k \le j} x_i \ge \binom{j-i+2}{2} \right\}$$

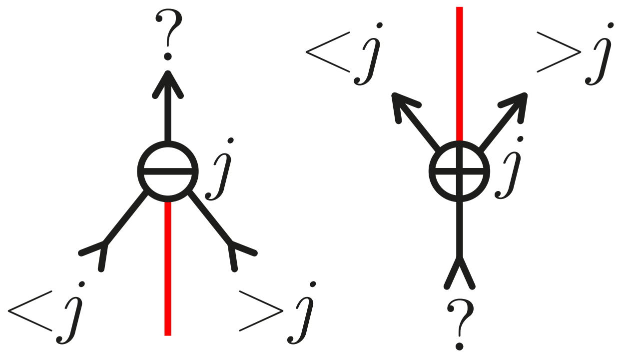
Loday, Realization of the Stasheff polytope ('04)



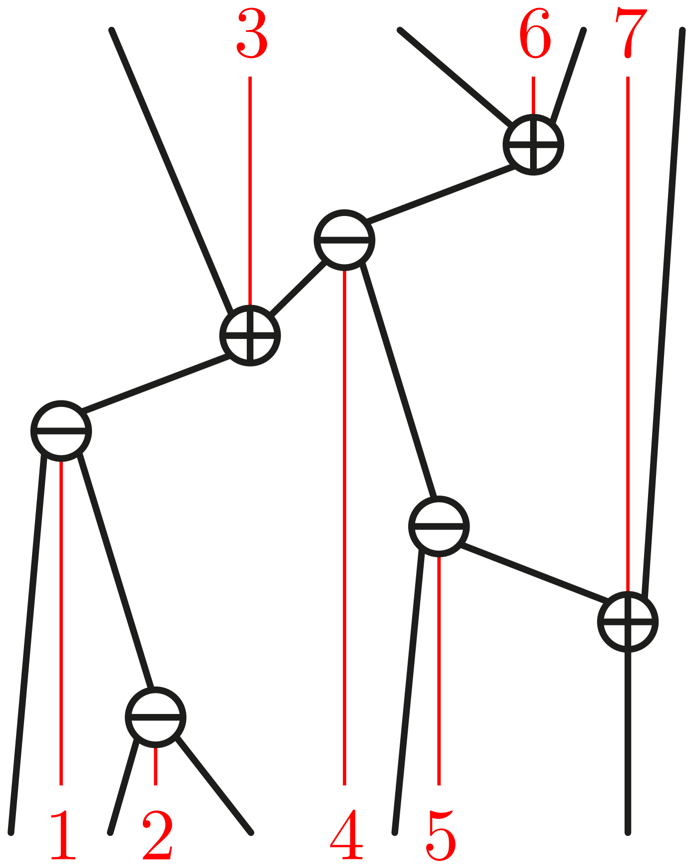


CAMBRIAN TREES

Cambrian tree $=$ directed and labeled (with $[n]$) trees with the following local rule



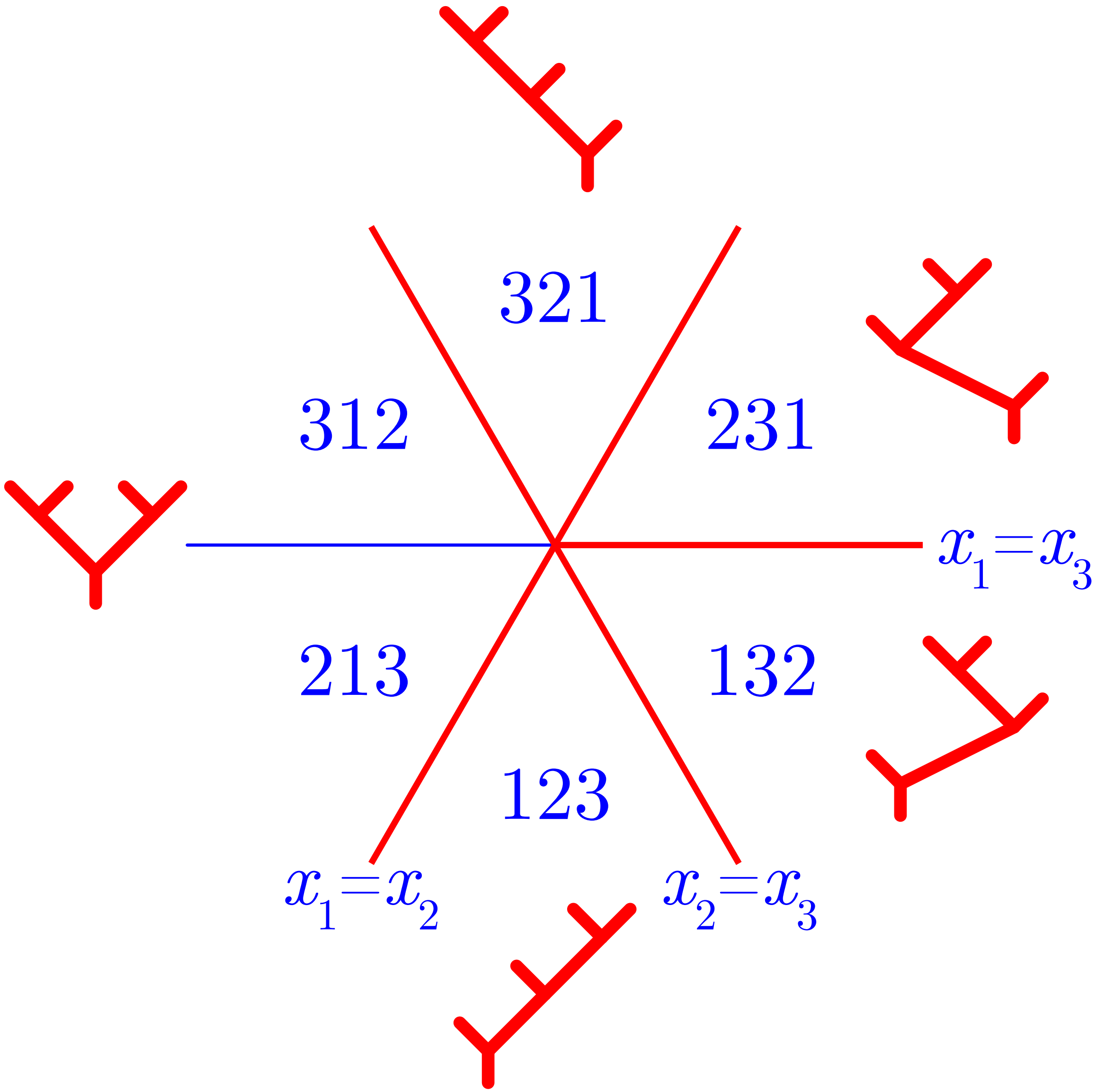
Rule for Cambrian trees



CAMBRIAN FANS

cone of a Cambrian tree T = $C(T) = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \leq x_j$ for each edge $i \to j$ in $T\}$

Cambrian fan = $\{C(T) \mid T$ binary tree on n nodes$\}$



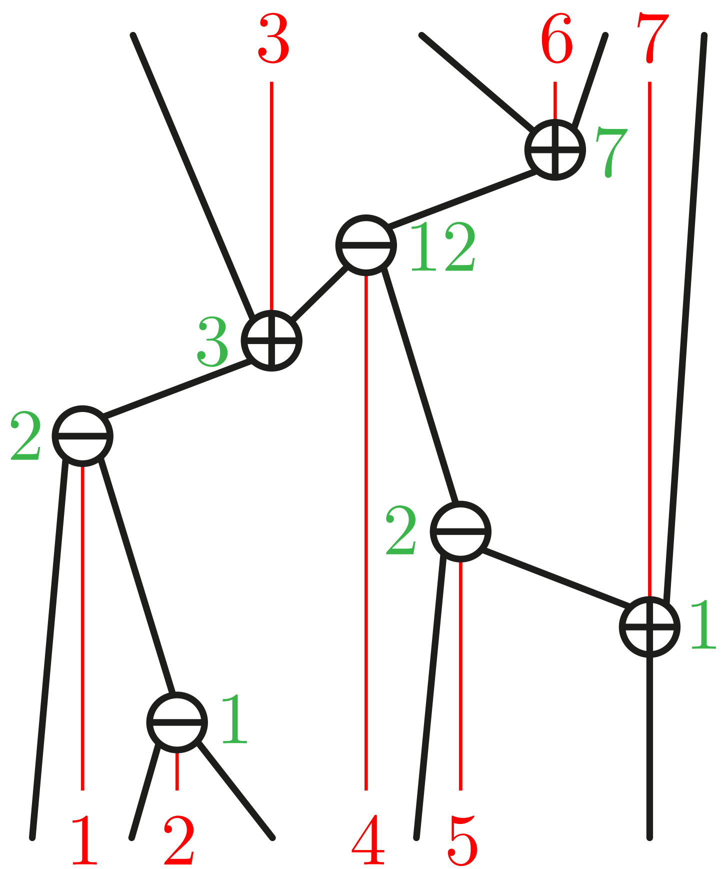
HOHLWEG-LANGE'S ASSOCIAHEDRA

For any signature $\varepsilon \in \pm^{n+1}$, $\qquad$ Asso$(\varepsilon) := \operatorname{conv}\{\mathbf{HL}(\mathrm{T}) \mid \mathrm{T}\ \varepsilon\text{-Cambrian tree}\}$

$$\text{with } \mathbf{HL}(\mathrm{T})_j := \begin{cases} \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = - \\ n + 2 - \ell(\mathrm{T}, j) \cdot r(\mathrm{T}, j) & \text{if } \varepsilon(j) = + \end{cases}$$

Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)

Lange-P., *Associahedra via spines* ('13^{+})



HOHLWEG-LANGE'S ASSOCIAHEDRA

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Hohlweg-Lange, *Realizations of the associahedron and cyclohedron* ('07)
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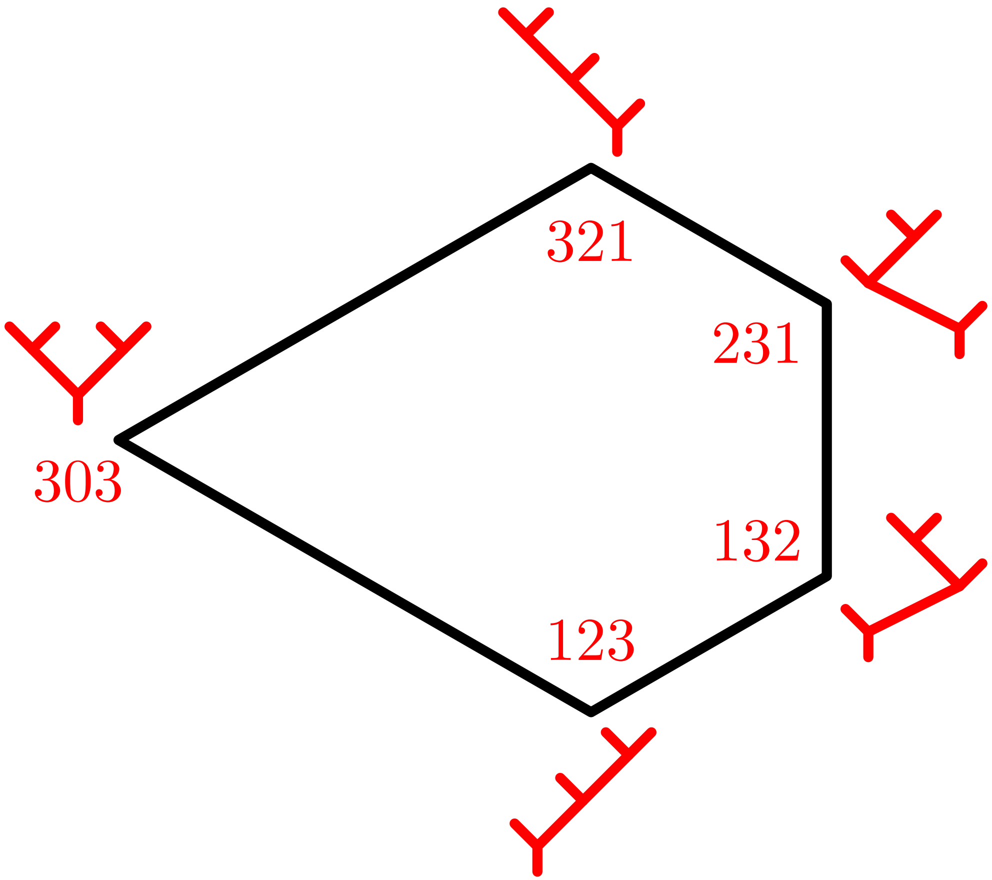

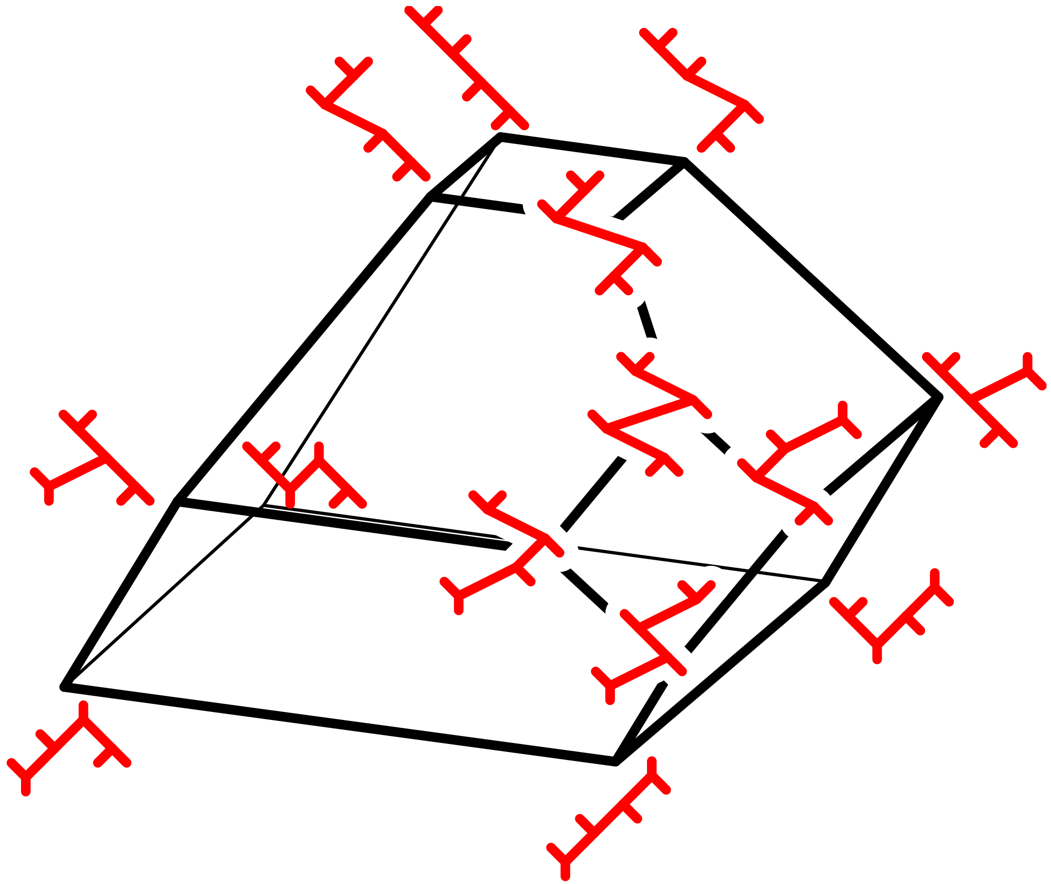

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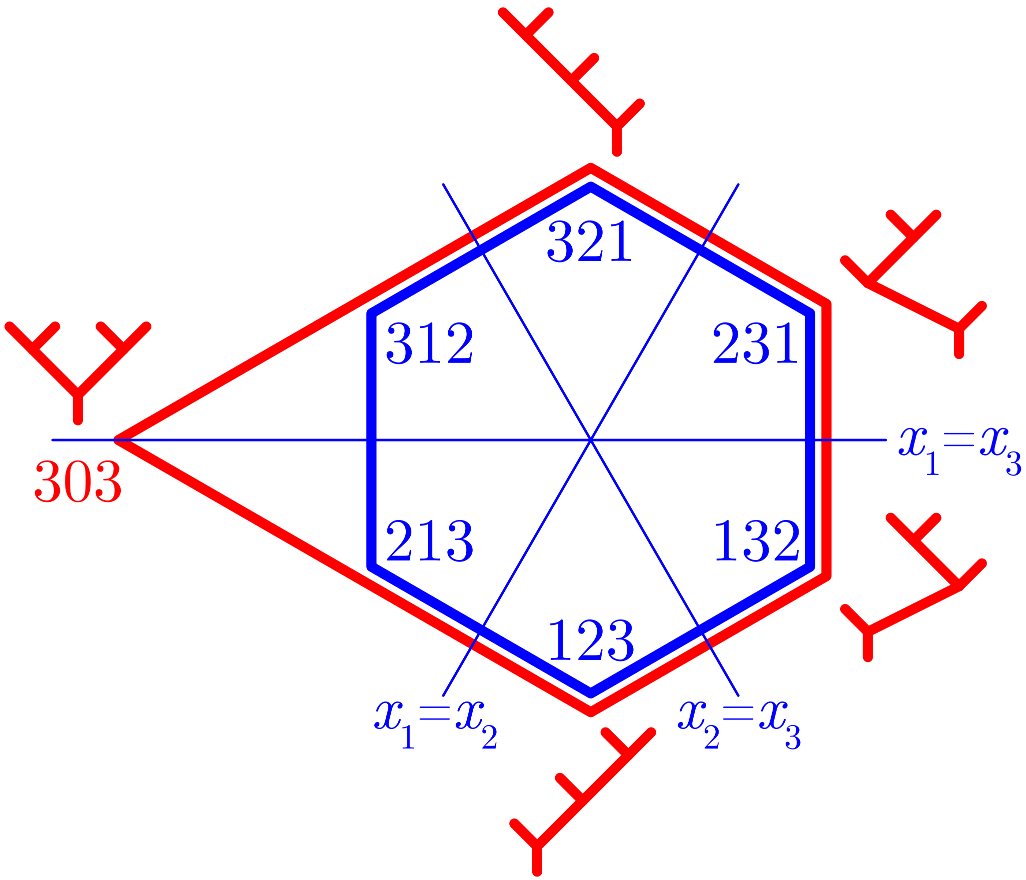
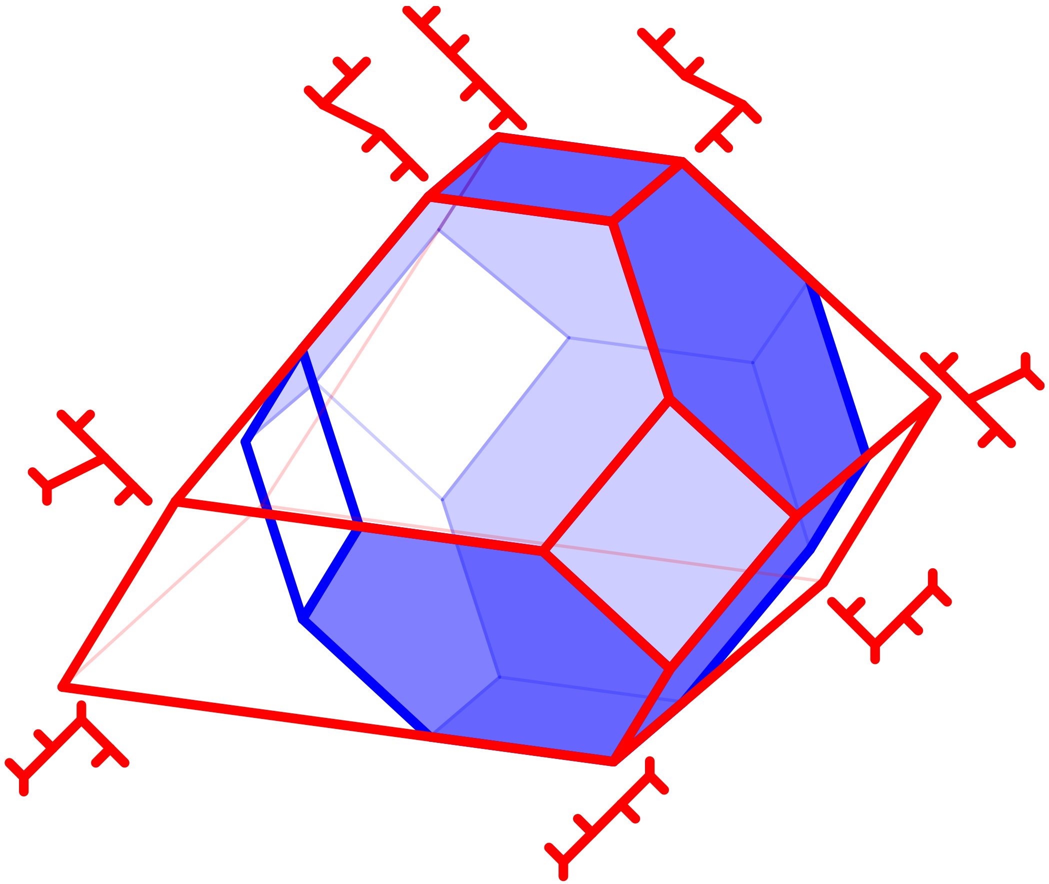




COMPATIBILITY FANS

COMPATIBILITY FANS

T° an initial triangulation

δ, δ' two internal diagonals

compatibility degree between δ and δ'

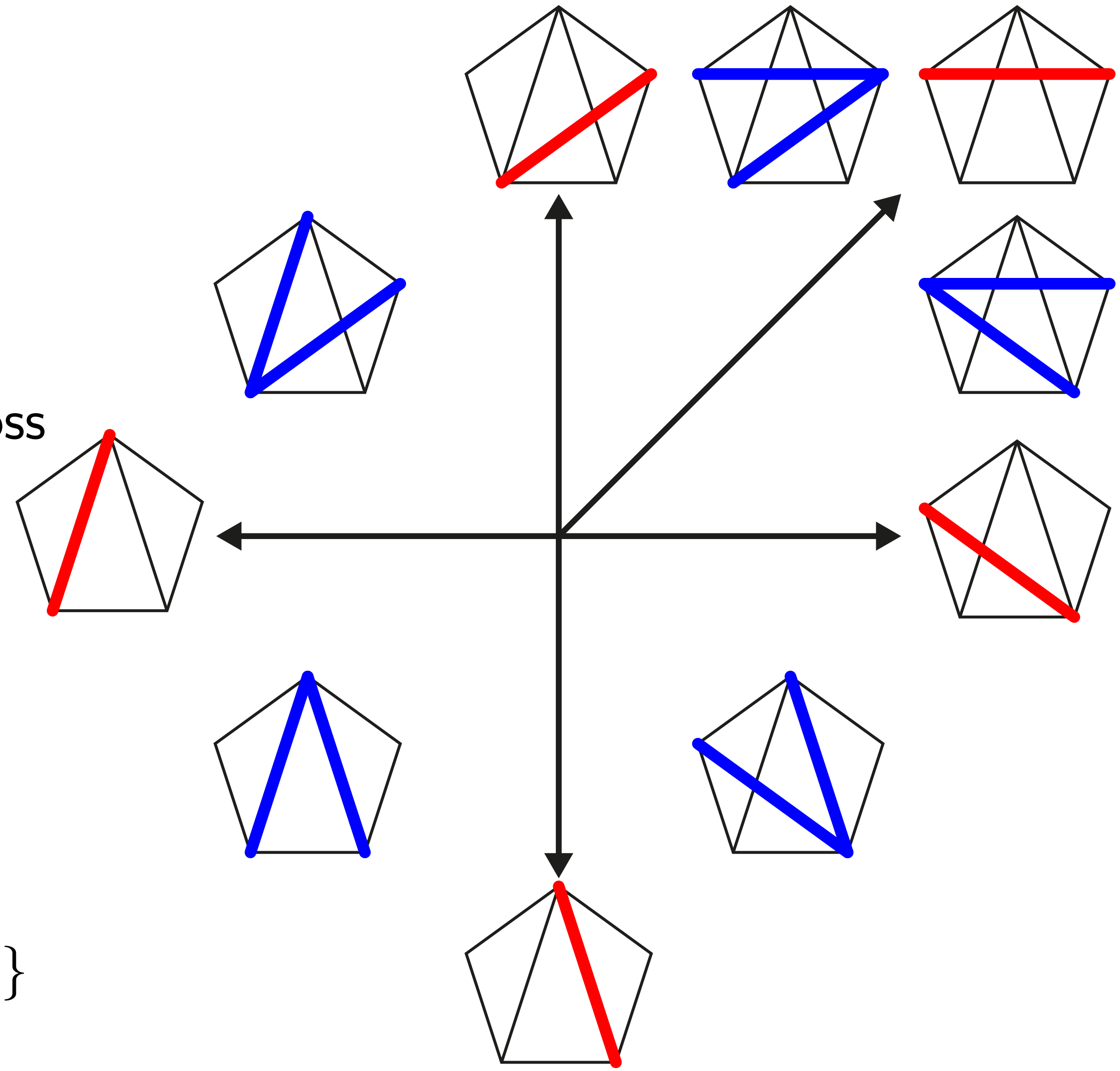
$$(\delta \parallel \delta') = \begin{cases} -1 & \text{if } \delta = \delta' \\ 0 & \text{if } \delta \text{ and } \delta' \text{ do not cross} \\ 1 & \text{if } \delta \text{ and } \delta' \text{ cross} \end{cases}$$

compatibility vector of δ wrt T°:

$$\mathbf{d}(\mathrm{T}^\circ, \delta) = \big[(\delta^\circ \parallel \delta)\big]_{\delta^\circ \in \mathrm{T}^\circ}$$

compatibility fan wrt T°

$$\mathcal{D}(\mathrm{T}^\circ) = \{\mathbb{R}_{\geq 0}\, \mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$$



Fomin-Zelevinsky, Y-Systems and generalized associahedra ('03)

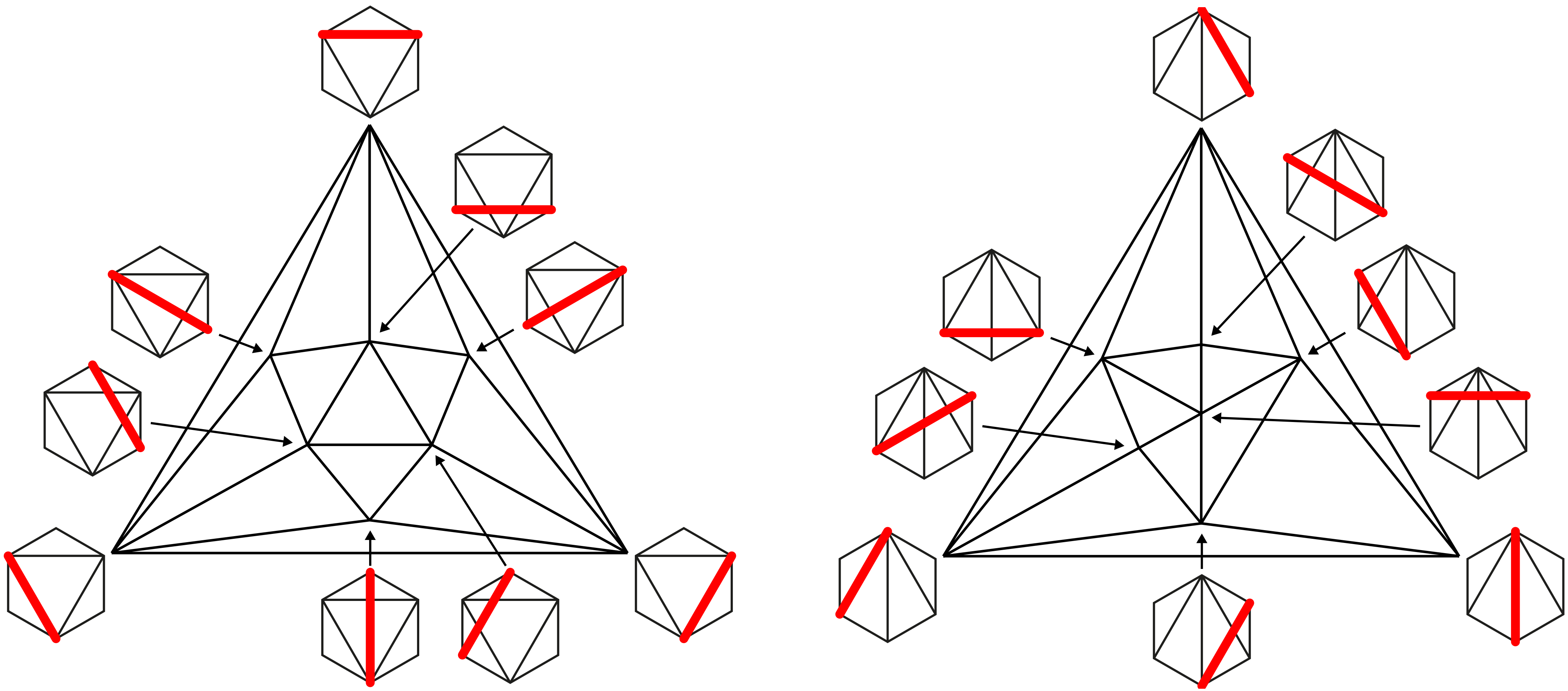
Fomin-Zelevinsky, Cluster algebras II: Finite type classification ('03)

Chapoton-Fomin-Zelevinsky, Polytopal realizations of generalized associahedra ('02)

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

COMPATIBILITY FANS

Different initial triangulations T° yield different realizations



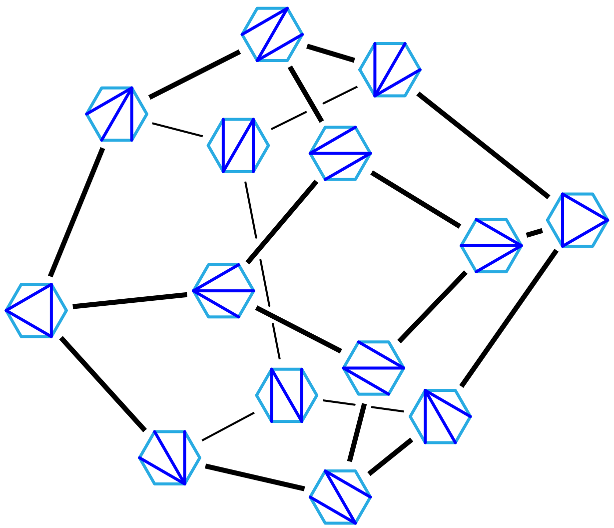
THM. For any initial triangulation T°, the cones $\{\mathbb{R}_{\geq 0}\,\mathbf{d}(\mathrm{T}^\circ, \mathrm{D}) \mid \mathrm{D} \text{ dissection}\}$ form a complete simplicial fan. Moreover, this fan is always polytopal.

Ceballos-Santos-Ziegler, Many non-equivalent realizations of the associahedron ('11)

WHAT SHOULD I TAKE HOME FROM THIS TALK?

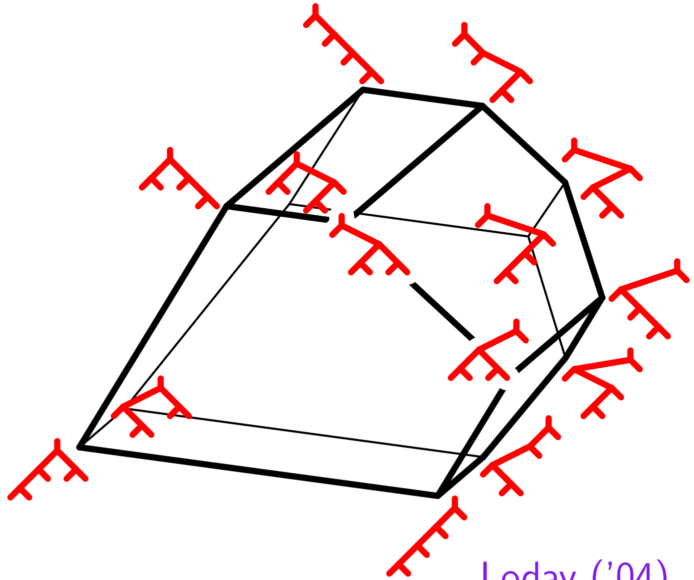
THREE FAMILIES OF REALIZATIONS

SECONDARY POLYTOPE



Gelfand-Kapranov-Zelevinsky ('94)
Billera-Filliman-Sturmfels ('90)

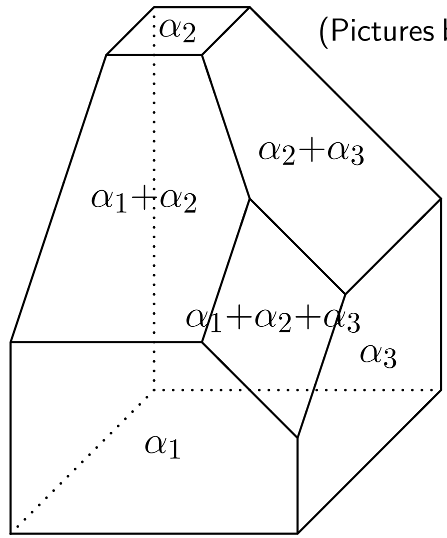
LODAY'S ASSOCIAHEDRON



Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)

CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON

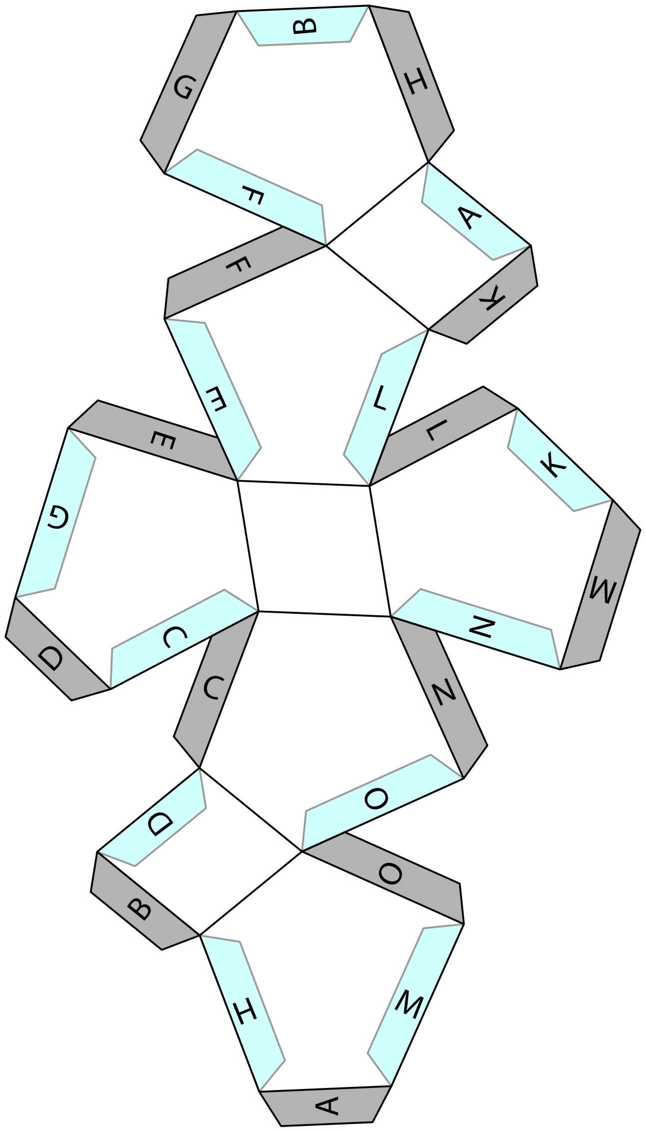
(Pictures by CFZ)



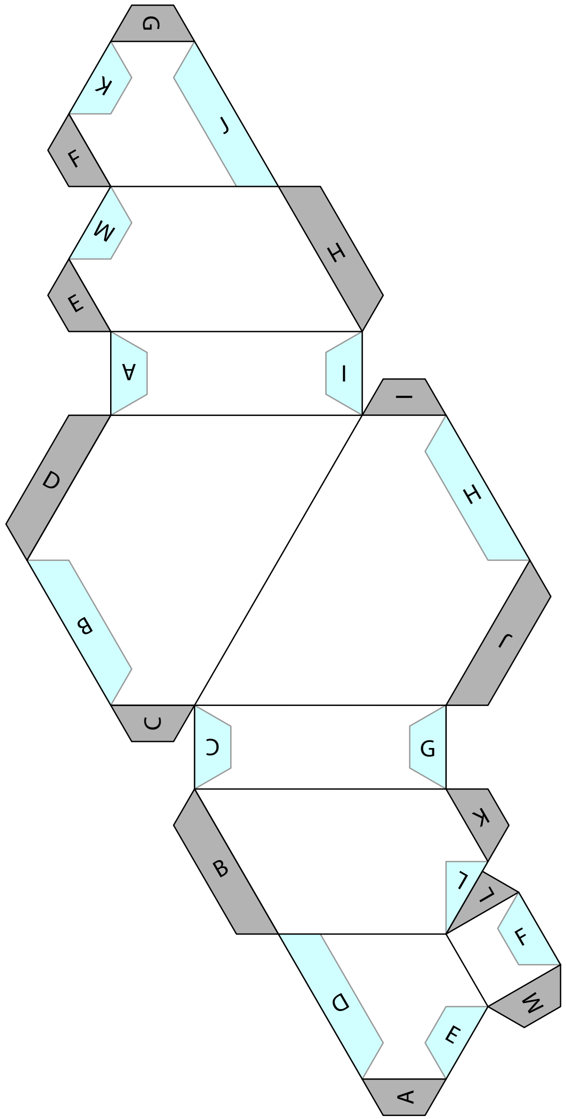
Chapoton-Fomin-Zelevinsky ('02)
Ceballos-Santos-Ziegler ('11)

TAKE HOME YOUR ASSOCIAHEDRA!

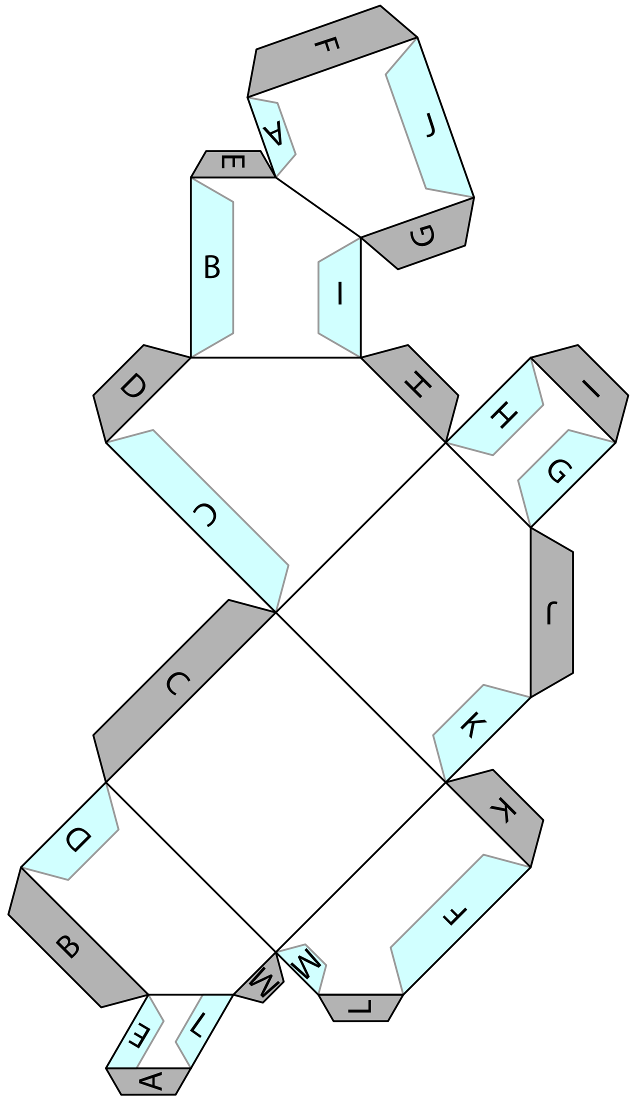
SECONDARY POLYTOPE



LODAY'S ASSOCIAHEDRON



CHAP.-FOM.-ZEL.'S ASSOCIAHEDRON



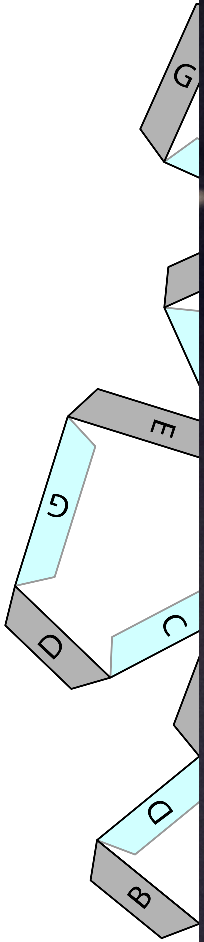
TAKE HOME YOUR ASSOCIAHEDRA!

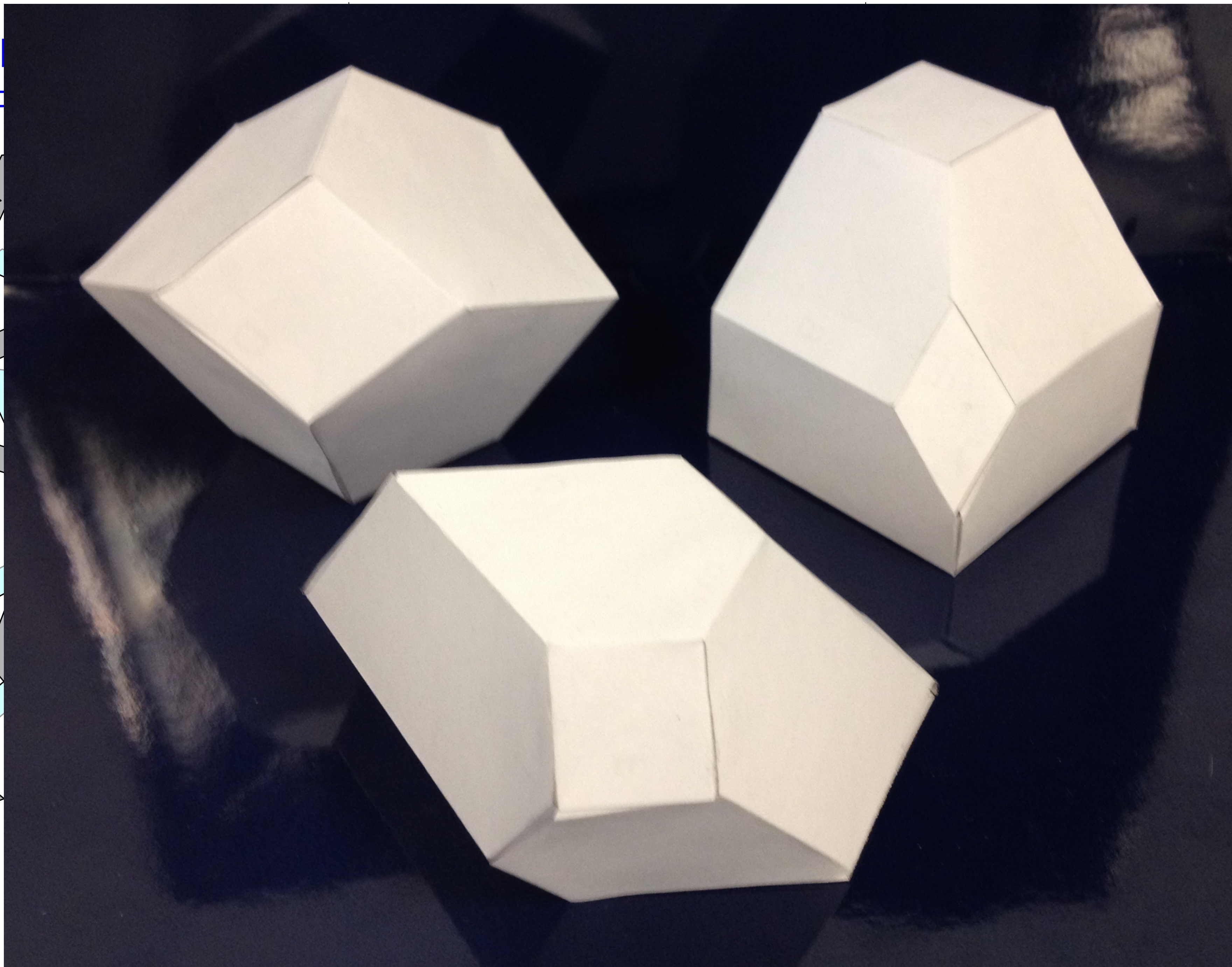
SI

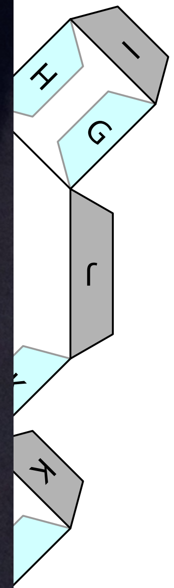
L.'S

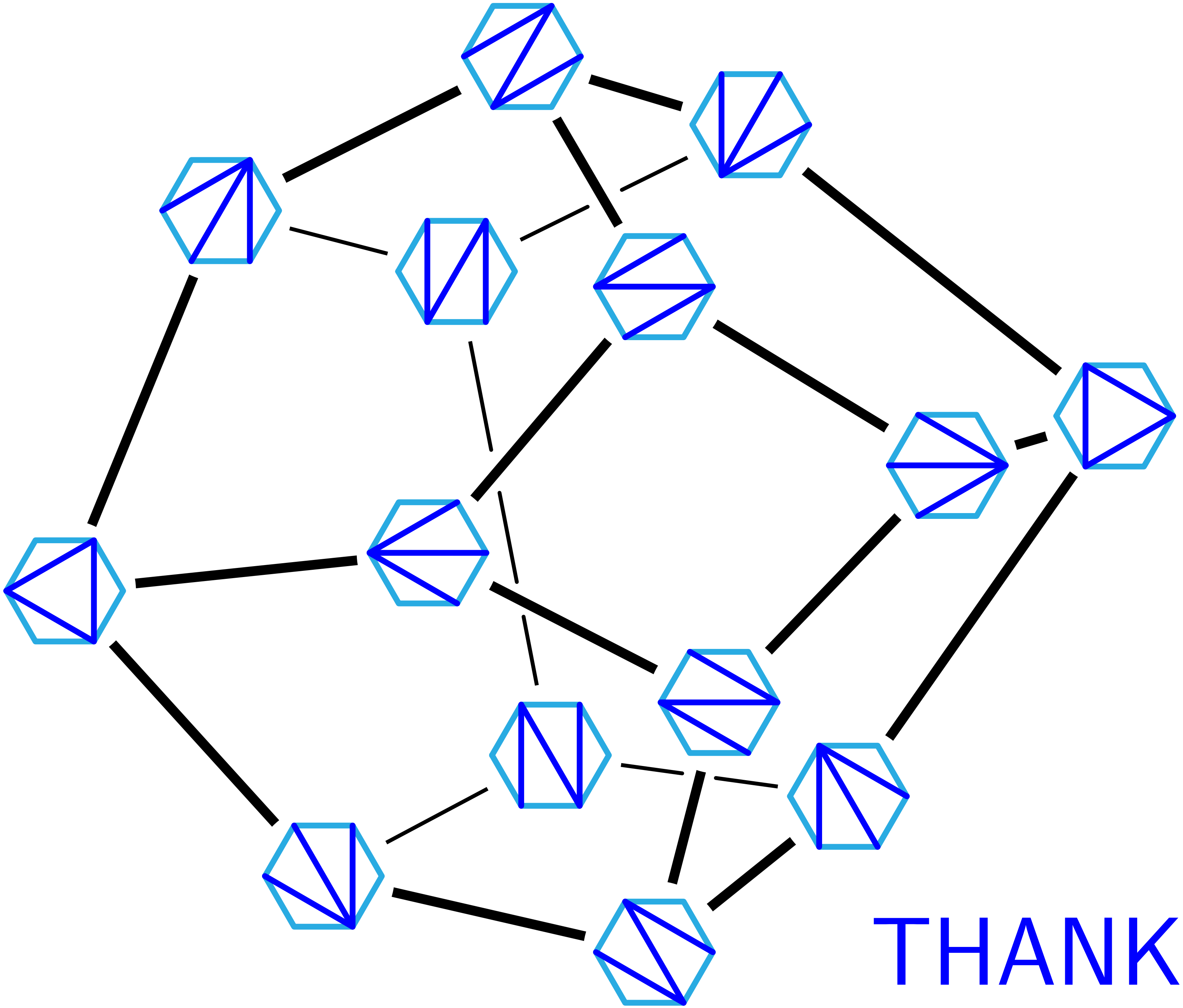
R

ON

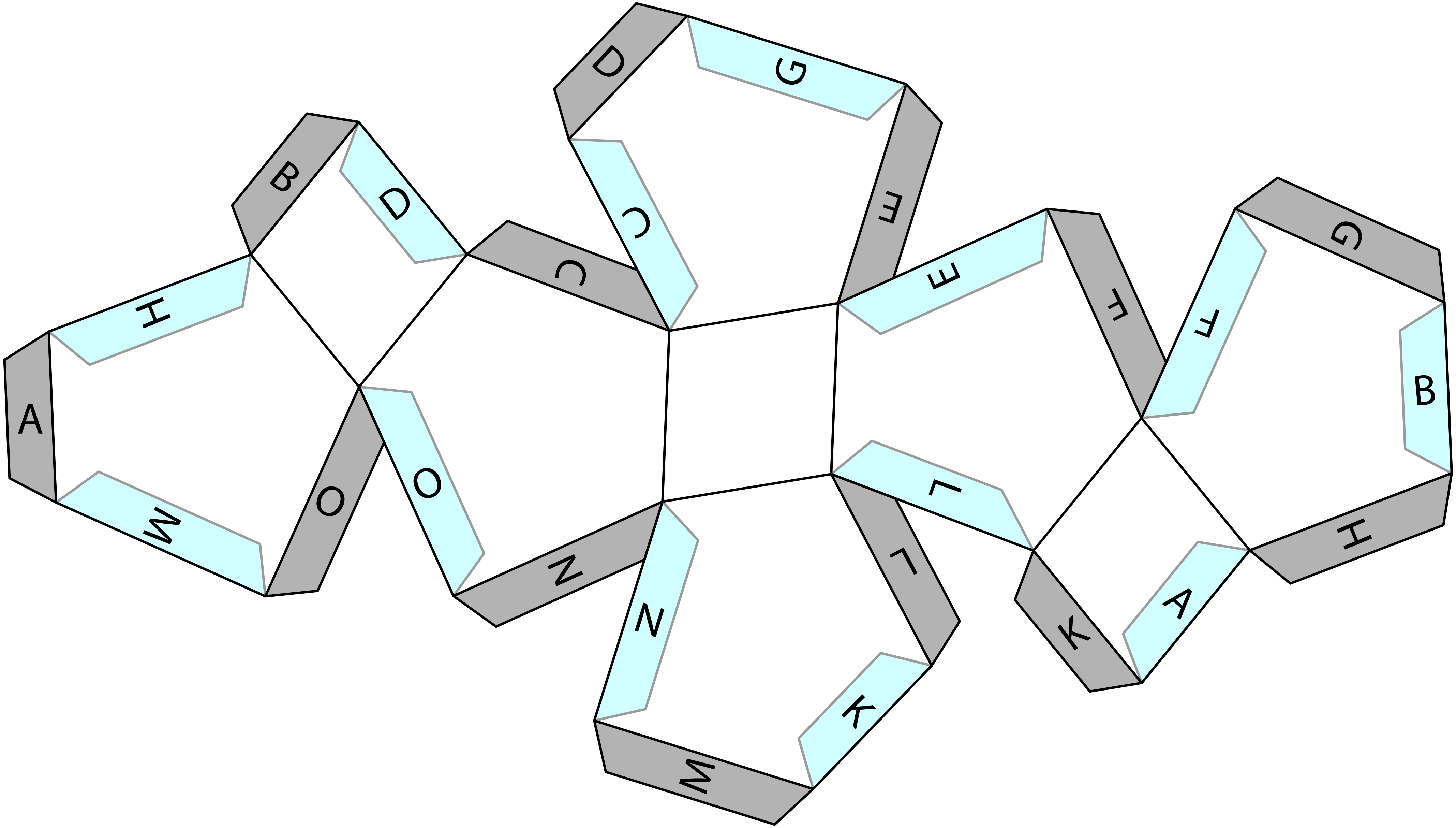








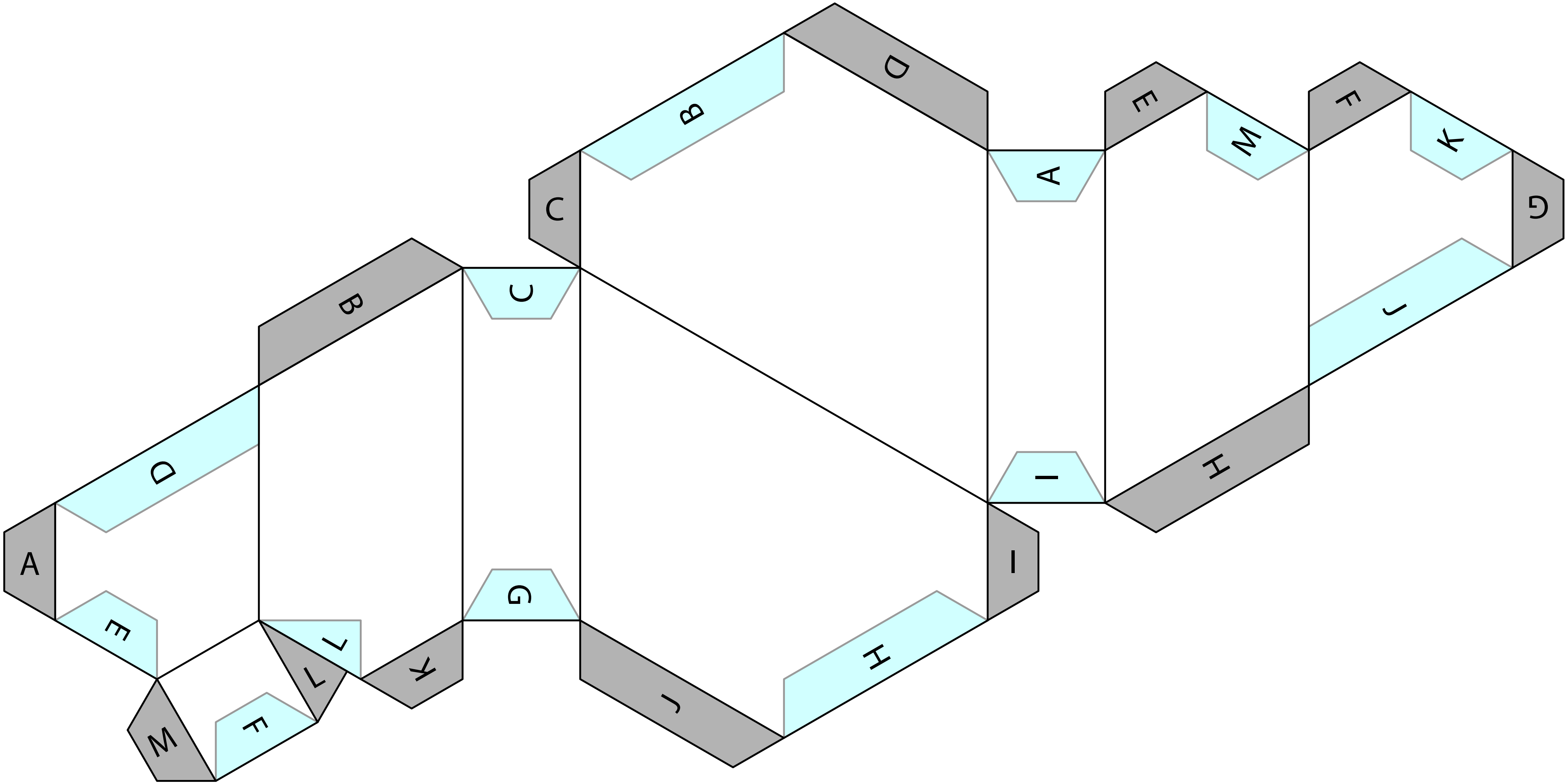
THANK YOU



SECONDARY POLYTOPE

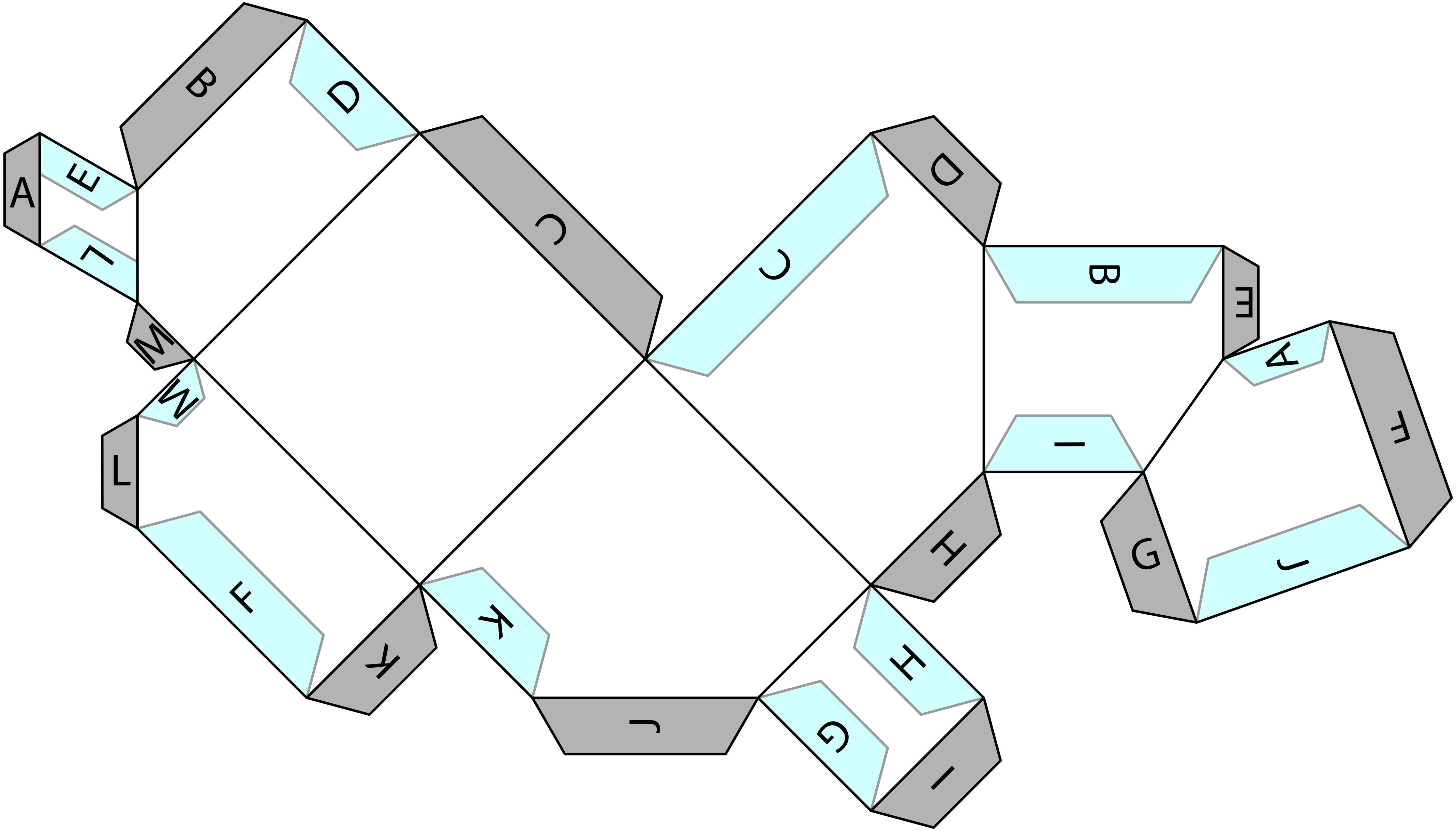
Gelfand-Kapranov-Zelevinsky ('94)

Billera-Filliman-Sturmfels ('90)



LODAY'S ASSOCIAHEDRON

Loday ('04)
Hohlweg-Lange ('07)
Hohlweg-Lange-Thomas ('12)



CHAPOTON-FOMIN-ZELEVINSKY'S ASSOCIAHEDRON

Chapoton-Fomin-Zelevinsky ('02)

Ceballos-Santos-Ziegler ('11)